Theory of Buildings and Applications in Number Theory

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- Buildings
- Applications
- Buildings in Higher Number Theory

Buildings

Reflection Groups

- Let V be finite-dimensional real vector space with an inner product, H is a hyperplane in V.
- The **reflection** with respect to H is the linear transformation $s_H : V \to V$ which is identity on H and is multiplication by -1 on the orthogonal complement $H^\perp$ of H.
- If α is a nonzero vector in $H^\perp$, so that $H = \alpha^\perp$, we will write s_α instead of s_H.

$$s_\alpha(x) = s_\alpha(h + \lambda\alpha) = h - \lambda\alpha = x - 2\lambda\alpha = x - 2\frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

- A finite reflection group is a finite group W of invertible linear transformations of V generated by reflections s_H, where H ranges over a set of hyperplanes.
- Finite reflection groups have been completely classified up to isomorphism:
  1. Type $A_n (n \geq 1)$;
  2. Type $C_n (n \geq 2)$ (This corresponds to root systems of type B_n and type C_n);
  3. Type $D_n (n \geq 4)$;
  4. Type E_6, E_7, E_8;
  5. Type F_4;
  6. Type G_2;
  7. Type H_3, H_4 (This doesn't correspond to any root system).

Chamber Complexes of Finite Reflection Groups

- Let $\mathcal{H} = \{H_i\}_{i \in I}$ be a family of hyperplanes in V. For each $i \in I$, suppose that $f_i : V \to \mathrm{R}$ is a nonzero linear function such that H_i is defined by $f_i = 0$.
- A cell in V with respect to the family $\mathcal{H}$ is a nonempty set A obtained by choosing for each $i \in I$, a sign $\sigma_i \in \{-, +, 0\}$, such that

$$A = \bigcap_{i \in I} U_i,$$

  1. $U_i = H_i = \{x \in V | f_i(x) = 0\}$, if $\sigma_i = 0$,
  2. $U_i = \{x \in V | f_i(x) > 0\}$, if $\sigma_i = +$,
  3. $U_i = \{x \in V | f_i(x) < 0\}$, if $\sigma_i = -$.
- The cells such that $\sigma_i \neq 0$ are called chambers. We let $\Sigma(\mathcal{H})$ denote the set of all cells and let $\mathcal{C}(\mathcal{H})$ be the subset of all chambers.

Let C be a fixed chamber, called the fundamental chamber, and let S be the set of reflections with respect to walls of C.

- The set S generates W.
- The action of W is simply transitive on the set $\mathcal{C}$ of chambers. Thus there is a $1-1$ correspondence between W and $\mathcal{C}$ defined by $w \leftrightarrow wC$.

Suppose that G is a group and S is a set of generators of G such that $S = S^{-1}$ and $e \notin S$. Then the **Cayley graph** of (G, S) is the graph whose vertex set is G and whose edges are the pairs (g, h) such that $h = gs$ for some $s \in S$.

- The chamber graph of $\Sigma(W, S)$ is isomorphic to the Cayley graph of (W, S).

Coxeter Groups

- A Coxeter group is a group W which has a generator set $S = \{r_1, r_2, \cdots\}$ such that W can be defined by
$$\langle r_1, r_2, \cdots | (r_i r_j)^{m_{ij}} = 1 \rangle.$$
where $m_{ii} = 1$ and $m_{ij} = m_{ji} \leq \infty$, if $i \neq j$. The pair (W, S) where W is a Coxeter group with generators $S = \{r_1, \cdots, \cdots\}$ is called a Coxeter system.
- For a Coxeter system (W, S), a standard coset in W is a coset of the form wW_J with $w \in W$ and $W_J := \langle J \rangle$ for some subset $J \subset S$.
- Let $\Sigma(W, S)$ be the poset of standard cosets in W, ordered by reverse inclusion. Thus $B \leq A$ in Σ if and only if $B \supseteq A$ as subsets of W, in which case we say that B is a face of A. We call $\Sigma(W, S)$ the Coxeter complex associated to (W, S).
- A simplicial complex Σ is called a Coxeter complex if it is isomorphic to $\Sigma(W, S)$ for some Coxeter system (W, S).

Buildings

Definition

A building is a simplicial complex Δ that can be expressed as the union of sub-complexes Σ (called apartments) satisfying the following axioms:

1. Each apartment Σ is a Coxter complex.
2. For any two simplicies $A, B \in \Delta$, there is an apartment containing both of them.
3. If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

Buildings as Chamber Complexes

- Let Δ be a finite dimensional simplicial complex. A **gallery** is a sequence of maximal simplicies in which any two consecutive one are **adjacent**, i.e., distinct and have a common codimension 1 face.
- We say that Δ is a **chamber complex** if all maximal simplicies have the same dimension and any two can be connected by a gallery.
- A **chamber** in a chamber complex is a maximal simplex. A codimension 1 face of a chamber will be called a **panel**.
- A chamber complex is said to be **thin** if each panel is a face of exactly two chambers.
- A chamber complex is said to be **thick** if each panel is a face of at least 3 chambers.
- For a chamber complex Δ, we let $\mathcal{C}(\Delta)$ denote the set of its chambers. Then there is well-defined **distance function** $\delta(-,-)$ on $\mathcal{C}(\Delta)$, which is defined to be the minimal length of the galleries which joined these two chambers.

Buildings as W-Metric Spaces

- A building of type (W, S) is as pair $(\mathcal{C}, \delta)$ consisting of a nonempty set $\mathcal{C}$, whose elements are called chambers, together with a map $\delta : \mathcal{C} \times \mathcal{C} \to W$, called the Weyl distance function, which satisfies the following conditions for all $C, D \in \mathcal{C}$:
  1. $\delta(C, D) = 1$ if and only if $C = D$.
  2. If $\delta(C, D) = w$ and $C' \in \mathcal{C}$ satisfies $\delta(C', C) = s \in S$, then $\delta(C', D) = sw$ or w. If in addition $l(sw) = l(w) + 1$, then $\delta(C', D) = sw$.
  3. If $\delta(C, D) = w$, then for any $s \in S$, there exists a chamber $C' \in \mathcal{C}$ such that $\delta'(C', C) = s$ and $\delta(C', D) = sw$.
- The axioms here is very similar to the axioms of metric spaces. We therefore sometimes call (C, δ) a W-metric space.

Metric Realizations of Buildings

- Let Z be a metric space with a family of nonempty subsets Z_s, $s \in S$. We can think Z as the model for a closed chamber, and Z_s be its s-panel.
- We define a equivalence relation on $\mathcal{C}(\Delta) \times Z$ by setting $(C, z) \simeq (D, z)$ if $\delta(C, D) = s$ and $z \in Z_s$. This is equivalent to say $(C, z) \sim (C', z')$ if and only if $z = z'$ and there is a gallery $C = C_0, \cdots, C_l = C'$ such that $\delta(C_{i-1}, C_i) = s_i \in S$ and $z \in Z_{s_i}$ for $i = 1, \cdots, l$.
- For a building Δ, we define the Z-realization of Δ, denoted as $X = X(Z, \Delta)$ to be the quotient of $\mathcal{C}(\Delta) \times Z$ by the equivalence relation defined above.
- There is a metric on X such that $\{C\} \times Z$ is isometric to Z.

Group Actions on Buildings

- We say an action of G on Δ is **strongly transitive** if G acts transitively on the sets of pairs (Σ, C) consisting of an apartment $\Sigma \in \Delta$ and a chamber $C \in \Sigma$.
- Assume that the G-action on Δ is strongly transitive, and we choose a pair (Σ, C). We will often refer to C as the fundamental chamber and to Σ as the fundamental apartment.
- We now define the three subgroups of G:

  $$B := \{g \in G | gC = C\},$$
  $$N := \{g \in G | g\Sigma = \Sigma\},$$
  $$T := \{g \in G | g \text{ fixes } \Sigma \text{ pointwise}\}.$$

  We fined that T is a normal subgroup of N, being the kernel of the homomorphism $f : N \to W$ induced by the action of N on Σ. We have $W \cong N/T$ since f is surjective.
- We say an action of G on a building Δ is **Weyl transitive** if for each $w \in W$, the action is transitive on the set of ordered pairs (C, D) of chambers with $\delta(C, D) = w$.

The Bruhat Decomposition

- Suppose that the action of G on Δ is Weyl transitive, and let B be the stabilizer of a chamber C. Then there is a bijection on $B \setminus G/B \to W$ given by $BgB \mapsto \delta(C, gC)$.
- Let G be group, $B \subset G$ a subgroup, (W, S) a Coxeter system, and $C : W \to B \setminus G/B$ be map of set. They satisfying the following axioms:
  1. $C(w) = B$ if and only if $w = 1$.
  2. $C : W \to B \setminus G/B$ is surjective, i.e., $G = \bigcup_{w \in W} C(w)$.
  3. For any $s \in S$ and $w \in W$,

$$C(sw) \subset C(s)C(w) \subset C(sw) \cup C(w).$$

- Standard parabolic subgroups: $P_J := \bigcup_{w \in W_J} C(w)$, $W_J = \langle J \rangle$, $J \subset S$.
- Given a Bruhat decomposition of (G, B), we denote by $\Delta(G, B)$ the poset of the standard parabolic subgroups, ordered by the reverse inclusion.
- $\Delta = \Delta(G, B)$ is a building, and the natural action of G on Δ by left multiplication is Weyl transitive.

BN-Pairs

Let B and N be two subgroups of a group G, we say B and N is a BN-pair if B and N generate G, the intersection $T := B \cap N$ is normal in N, and the quotient $W := N/T$ admits a set of generators S such that the following two conditions hold:

- For $s \in S$ and $w \in W, sBw \subseteq BswB \cup BwB$.
- For $s \in S, sBs^{-1} \nleq B$.

The group W will be called the Weyl group associated to the BN-pair. We also say that the quadruple (G, B, N, S) is a Tits system.

- Suppose that we have BN-pair in G, then the generating set S is uniquely determined, and (W, S) is a Coxeter system. There is a thick building $\Delta = \Delta(G, B)$ that admits a strongly transitive G-action, such that B is a stabilizer of a fundamental chamber and N stabilizers a fundamental apartment and is transitive on its chambers.

Applications

Buildings Associated to Vector Spaces

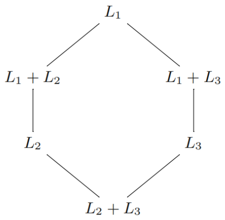
- Let V be a vector space of dimension $n \geq 2$ over an arbitrary field.
- We let $\Delta = \Delta(V)$ be the flags of V, which are chains

$$V_1 \subset V_2 \subset \cdots \subset V_k$$

  of nonzero proper subspaces of V.
- The maximal simplices of Δ are the chains

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1}$$

  with $\dim V_i = i$.
- An apartment when $n = 3$



Building of Semisimple Algebraic Groups

Suppose that G is a semisimple algebraic group over an arbitrary field k. We construct a building $\Delta(G)$ by the Tits system (G, B, N, S) related to G.

- If k is algebraically closed. We take B to be a Borel subgroup of G, T a maximal torus T contained in B, and let $N = N_G(T)$.
  1. The vertices of $\Delta(G)$ are maximal proper parabolic subgroups of G.
  2. Vertices $P_1, \cdots, P_m$ forms the vertices of a simplex σ if and only if the intersection $P_1 \cap \cdots \cap P_m$ is also a parabolic subgroup, which corresponds to the simplex σ.
  3. The chambers of $\Delta(G)$ correspond to the Borel subgroups of G.
  4. The apartments of $\Delta(G)$ correspond to the maximal tori of G.
- If k is not algebraically closed. We take B to be a minimal parabolic subgroup of G, T a maximal split torus contained in B, and let $N = N_G(T)$.
  1. The vertices of $\Delta(G)$ are maximal proper parabolic subgroups of G.
  2. Vertices $P_1, \cdots, P_m$ forms the vertices of a simplex σ if and only if the intersection $P_1 \cap \cdots \cap P_m$ is also a parabolic subgroup, which corresponds to the simplex σ.
  3. The chambers of $\Delta(G)$ correspond to the minimal parabolic subgroups of G.
  4. The apartments of $\Delta(G)$ correspond to the maximal split tori of G.

Buildings of Reductive Groups over Local Fields

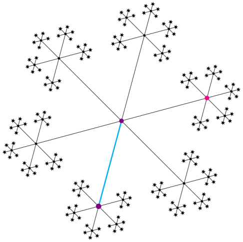
- G be a connected reductive group over a nonarchimedean locally compact field F. The apartments of $\Delta(G)$ are

$$A = (X_*(S)/X_*(\mathbb{C})) \otimes \mathrm{R}$$

  from maximal split tori of G.
- We define a equivalence relation $\sim$ on the set $G \times A$ by $(g, x) \sim (h, y)$ if there exists an $n \in N_G(S)$ such that $nx = y$ and $g^{-1}h_n \in U_x$. The set of equivalence classes is defined by

$$X := G \times A/\sim .$$

  We call X the **Bruhat-Tits building** of G.
- The Bruhat-Tits Building of $\mathrm{SL}_2(\mathbb{Q}_5)$



Representations of p-Adic Reductive Groups

- Suppose that K is a compact open subgroup of G. The Hecke-algebra $\mathcal{H}(G, K)$ is the algebra of complex-valued functions f on G satisfying the following conditions:
  1. $f(kgk') = f(g)$ for any $g \in G$ and $k, k' \in K$;
  2. f is zero outside of a union of a finite number of KgK.
- The Hecke-algebra of G is defined to be

$$\mathcal{H}(G) = \cup_K \mathcal{H}(G, K),$$

  where K runs through a neighbourhood basis of 1 which consists of compact open subgroups of G.
- There is a bijection between the set of isomorphism classes of irreducible smooth representations of G and the set of isomorphism classes of non-degenerate simple $\mathcal{H}(G)$-module.
- In general, for a compact open subgroup I of G, we have the following adjunction:

$$(\cdot)^I : \mathrm{Rep}_R^\infty(G) \leftrightarrows \mathrm{Mod}_H : \mathrm{Ind}_I^G(R) \otimes_H (\cdot).$$

  When $R = \mathbb{C}$ and I is the pro-p Iwahori subgroup, then the above functors give an equivalence between the category Mod_H and the full subcategory Rep_G^I of $\mathrm{Rep}_R^\infty(G)$.

Suppose that $\mathcal{X}$ is a building. A coefficient system on a building $\mathcal{X}$ is a family $(\mathcal{F}_F, (r^F_{F'})_{F' \subset \bar{F}})$ indexed by the faces of $\mathcal{X}$, where $\mathcal{F}_F \in \mathrm{Mod}_R$ and for any faces F and F' such that $F' \subset \bar{F}$, there is a R-module map $r^F_{F'} : \mathcal{F}_F \to \mathcal{F}_{F'}$. These morphisms must satisfying the following two properties:

$$f^F_F = \mathrm{Id}_{\mathcal{F}_F} \quad \text{and} \quad r^F_{F''} = r^{F'}_{F''} \circ r^F_{F'},$$

for any faces $F'' \subseteq \overline{F'} \subseteq \overline{F}$ of $\mathcal{X}$.

An object $\mathcal{F} \in \mathrm{Coeff}(\mathcal{X})$ is called G-equivariant if it comes with a family of coefficient systems $\{c_g : \mathcal{F} \to g_* \mathcal{F}\}_{g \in G}$ such that

1. $c_1 = \mathrm{Id}_{\mathcal{F}}$,
2. $c_{gh} h_* c_g \circ c_h, \forall g, h \in G$.

For a coefficient system $\mathcal{F}$, we define a G-equivariant sheaf $\mathbb{S}(\mathcal{F})$ as follows: For any open subset $\Omega \subseteq \mathcal{X}$, we define

$$\mathbb{S}(\mathcal{F})(\Omega) := \{s : \Omega \to \coprod_{z \in \Omega} \mathcal{F}^*_{F_z} \text{ satisfying}\}$$

1. $\forall z \in \Omega, s(z) \in \mathcal{F}^*_{F_z}$,
2. $\forall z \in \Omega, \exists V \subset \Omega \mathrm{Star}(F_z)$ open neighborhood of z such that $\forall z' \in V, s(z') = (r^{F_{z'}}_{F_z})^*(s(z))$.

We define a functor

$$\begin{aligned} \mathcal{F} : \qquad & \mathrm{Mod}_H \to \mathrm{Coeff}_G(\mathcal{X}) \\ & M \mapsto \mathcal{F}(M), \end{aligned}$$

where $\mathcal{F}(M)_F = \mathrm{Im}(X^{I_F} \otimes_H M \xrightarrow{\tau_{M,F}} \mathrm{Hom}_H(\mathrm{Hom}_H(X^{I_F}, H), M))$.

The functor

$$\mathbb{S} \circ \mathcal{F}(.) : \mathrm{Mod}_H \to \mathrm{Shv}_G(\mathcal{X})$$

is fully faithful. The essential image of the functor $\mathbb{S} : \mathrm{Coeff}_G^{fg}(\mathcal{X}) \to \mathrm{Shv}_G(\mathcal{X})$ is the full subcategory of constructible G-equivariant sheaves on $\mathcal{X}$ (Schneider-Stuhler, Kohlhaase).

Buildings in Higher Number Theory

Higher Dimensional Local Fields

Definition

- A 0-dimensional local field is a finite field.
- For $n \geq 1$, a n-dimensional local field is a complete discrete valuation field whose residue field is a $(n-1)$-dimensional local field.

One-dimensional local fields

1. $\mathbb{R}$, $\mathbb{C}$;
2. $\mathbb{F}_q((t))$;
3. Finite extension of $\mathbb{Q}_p$.

Two-dimensional local fields

1. $\mathbb{F}_q((t_1))((t_2))$;
2. $E((t))$ over a local nonarchimedean field E;
3. $E((t))$ over a local archimedean field E;
4. Finite extensions of $\mathbb{Q}_p\{\{t\}\}$.

Classification of n-Dimensional Local fields

Let F be a n-dimensional local field.

1. If $\mathrm{char} F \neq 0$ then
$$F \cong F^{(n)}((t_1)) \cdots ((t_n)),$$
where $F^{(n)}$ is a finite field.
2. If $\mathrm{char} F^{(n-1)} = 0$ then
$$F \cong F^{(n-1)}((t_1)) \cdots ((t_{n-1})),$$
where $F^{(n-1)}$ is a one-dimensional local field of characteristic 0.
3. In the remaining case, Let $2 \leq r \leq n$ be the unique integer such that $\mathrm{char} F^{(n-r)} = 0 \neq \mathrm{char} F^{(n+1-r)}$. Then F is isomorphic to a finite extension of
$$\mathbb{Q}\{\{t_1\}\} \cdots \{\{t_{r-1}\}\}((t_{r+1})) \cdots ((t_n))$$
where $\mathbb{Q}_q$ is the unramified extension of $\mathbb{Q}_p$ with residue field of $F^{(n)}$.

Rings of Integral Elements in Higher Local Fields

- If $n = 0$, we define $\mathcal{O}_F^0 = F$.
  If $n \geq 0$, we define $\mathcal{O}_F^n := \{x \in \mathcal{O}_F : \overline{x} \in \mathcal{O}_{\overline{F}}^{(n-1)}\}$, where $\mathcal{O}_{\overline{F}}^{(n-1)}$ is the rank $n-1$ ring of integers of $\overline{F}$, a field of discrete valuation of dimension $\geq n-1$.

$$F \supset \mathcal{O}_F = \mathcal{O}_F^{(1)} \supset \mathcal{O}_F^{(2)} \supset \cdots \supset \mathcal{O}_F^{(n)}.$$

- Suppose that F is a complete valuation field, and $t \in F$ is uniformizer, then we have

$$F = \mathcal{O}_F[t^{-1}], \quad F^\times \cong \mathcal{O}_F^\times \times t^{\mathbb{Z}},$$

  where $t^{\mathbb{Z}}$ denote the infinite cyclic group of $F^\times$ generated by t.
- A sequence of n-local parameters $t_1, \cdots, t_n \in F$ is a sequence of elements satisfying:
  1. t_n is a uniformizer of F.
  2. The reduction $\overline{t_1}, \cdots, \overline{t_{n-1}}$ of $t_1, \cdots, t_{n-1}$ form a sequence of local parameter for the filed $\bar{F}$ of dimension $n-1$.
- Suppose that F is a n-dimensional, then we have

$$F = O_F^{(n)}[t_1^{-1}, \cdots, t_n^{-1}], \text{ and } F^\times \cong (O_F^n)^\times \times t_1^{\mathbb{Z}} \times \cdots \times t_n^{\mathbb{Z}}.$$

Higher Adelic Spaces

Let $B = \mathrm{Spec}\mathcal{O}_K$ for a number filed K and let $\phi : X \to B$ be a B-scheme satisfying the following conditions:

- X is integral, regular and dimension 2.
- ϕ is proper and flat.
- The generic fibre X_K is a geometrically integral, smooth, projective curve over K. Fix a closed point $x \in X$, and a curve $y \subset X$, such that $x \in y \subset X$.
  1. $p_{y,x} = \ker(\mathcal{O}_{X,x} \to \mathcal{O}_{y,x})$.
  2. $\phi : \mathrm{Spec}\widehat{\mathcal{O}_{X,x}} \to \mathrm{Spec}\mathcal{O}_{X,x}$.
  3. For any $q \in \mathrm{Spec}\widehat{\mathcal{O}_{X,x}}$, such that $q \cap \mathcal{O}_{X,x} = p_{y,x}$, we call this q a local branches of y at x, denote all this q as $y(x)$.
- $K_{x,q} := \mathrm{Frac}(\widehat{(\mathcal{O}_x)_q})$ is a two dimensional local field.

$$K_{x,y} := \prod_{q \in y(x)} K_{x,q}, \quad \mathcal{O}_{x,y} := \prod_{q \in y(x)} \mathcal{O}_{x,q}$$

$$E_{x,y} := \prod_{q \in y(x)} E_{x,q}, \text{ where } E_{x,q} \text{ is the residue field of } K_{x,q}.$$

$$k_{x,y} := \prod_{q \in y(x)} k_{x,q}, \text{ where } k_{x,q} \text{ is the residue field of } E_{x,q}.$$

- We define a ring $\mathbb{A}_y$ as the restricted product of the rings $K_{x,y}$ for y fixed and $x \in y$,

$$\mathbb{A}_y = \prod_{x \in y}{}' K_{x,y}.$$

- We define the two-dimensional adele,

$$\mathbf{A}_X = \prod_{x \in y, y \subset X}{}'' K_{x,y}.$$

Automorphic Forms on Two-dimensional Adelic Spaces

For a reductive group G, let

$$\mathbb{T}_G = G(\mathbb{A}) \times G(\mathbb{A})/V(G(\mathbb{A}) \times G(\mathbb{A})).$$

Let K_G be the image of the map

$$G(\mathbb{B}) \times G(K) \to G(\mathbb{A}) \times G(\mathbf{A})/G(\mathbf{OA}) \to \mathbb{T}_G$$

Irreducible representations of $G \times G$ in the space of continuous $\mathbb{C}((X))$-valued functions on $\mathbb{T}_G/\mathbb{K}_G$, satisfying some restrictions, could be viewed as a candidate for unramified automorphic functions associated to G.

Buildings associated with Higher Local Fields (Parshin)

Let K be a n-dimensional local field, and let $\mathcal{O}' = \mathcal{O}_K^{(n)}$. Suppose that V is a vector space of dimension $l+1$ over K. We will define the building of the group $PGL(V)$.

$$\Delta(G, K/\cdots/k) = \bigcup_{0\leq m\leq n} \Delta_\bullet[m]$$

where for $n = 0$,

$$\Delta_0[0] = \{\text{nonzero proper subspaces } L \subset V\},$$

and for $n > 0$

$$\Delta_0[m] = \bigcup_{\text{Conditions}} \{\langle L\rangle | L \cong \mathcal{O}_{i_1} \oplus \cdots \oplus \mathcal{O}_{i_{l+1}} \text{ as } \mathcal{O}'\text{-modules}\}.$$

Here $\langle L\rangle$ is a class of $\mathcal{O}'$-module L up to aL, $a \in K^*$, and the conditions means that the union is over all

$$0 \leq i_{l+1} \leq \cdots, \leq i_1 = n \text{ for all } 1 \leq k \leq l+1,\, i_k = \text{ either n or m.}$$

For $i > 0$, we define

$$\Delta_i[m] = \left\{ \begin{array}{c} (\langle L_0\rangle, \cdots, \langle L_{i+1}\rangle) | \langle L_0\rangle, \cdots, \langle L_{i+1}\rangle \in \Delta_0[m]. \\ \text{and belongs to a maximal chain of } \mathcal{O}'\text{-submodules.} \end{array} \right\}$$

A set $\{L_\alpha\}_{\alpha\in I}$ of $\mathcal{O}'$-modules in V is called a chain if

1. for every $\alpha \in I$ and every $a \in K^*$, there exits α such that $aL_\alpha = L_{\alpha'}$,
2. the set $\{L_\alpha\}_{\alpha\in I}$ is totally ordered by the inclusion.

The vertices of $\mathrm{PGL}(2)$ over a 2-dimensional local field K.

$\Delta_0[2]$	22	$\langle O \oplus O\rangle$
$\Delta_0[1]$	21	$\langle O \oplus \mathcal{O}\rangle$
$\Delta_0[0]$	20	$\langle O \oplus K\rangle$

The vertices of $\mathrm{PGL}(3)$ over a 2-dimensional local field K.

$\Delta_0[2]$	222	$\langle O \oplus O \oplus O\rangle$
$\Delta_0[1]$	221	$\langle O \oplus O \oplus \mathcal{O}\rangle$
	211	$\langle O \oplus \mathcal{O} \oplus \mathcal{O}\rangle$
$\Delta_0[0]$	220	$\langle O \oplus O \oplus K\rangle$
	200	$\langle O \oplus K \oplus K\rangle$

- Suppose that S is a smooth projective algebraic surface over $\mathbb{F}_q$. We fix a point $P \in S$ and an irreducible curve C such that $P \in C$.
- (Parshin) Let $\mathcal{M}^\circ \subset \mathcal{M} = \mathrm{Bun}_G$ be the moduli space of $\mathcal{O}_S$-module F, such that F is trivial rank 2 on $X \setminus C$. Let $\mathcal{M}^{\circ\circ} \subset \mathcal{M}^\circ$ be the subspace corresponds to those which are locally trivial outside P. Then we have a map

$$\Psi : \mathcal{M}^{\circ\circ} \to \Delta_0(S, C, P)[2].$$

- Let $V \in \mathcal{M}$ and $W \subset V_C$, $\mathcal{A}_C(V, W)$ denote the subsheaf of V consisting of sections whose value at C is contained in W. To any vector bundle L on C, we define the Hecke operators :

$$\begin{aligned} T_L \quad &: \quad \mathbb{C}[\mathcal{M}] \to \mathbb{C}[\mathcal{M}] \\ & \quad f \mapsto T_L(f)(V \mapsto \sum_{W \subset V_C, (V|_C)/W \simeq L} f(\mathcal{A}_C(V, W))), \end{aligned}$$

Types of C	Hecke Algebras
$\mathbb{P}^1$	quantum algebra $U_q(sl_2)\|_{q=p^{1/2}}$ generated by E, F, K
elliptic curves	quantum toroidal algebra $U_q(\mathfrak{g}_{tor})\|_{q=p^{1/2}}$ generated by E_i, F_i, K_i

(Ginzburg-Kapranov) Langlands conjecture for surfaces.

1. For any pair (S, C), there is a natural algebraic homomorphism

$$\mathbb{C}[\mathrm{LocSys}_C] \to \mathrm{End}\mathbb{C}[\mathcal{M}^\circ], \quad f \mapsto T_f.$$

2. To any local system ϕ on S, one can associate an automotphic function $F_\phi \in \mathbb{C}[\mathcal{M}^\circ]$ such that

$$T_f(F_\phi) = f(\phi|_C) \cdot F_\phi, \forall f \in \mathbb{C}[\mathrm{LocSys}_C].$$

Global Langlands conjecture for surfaces.

1. Let $\mathfrak{B}$ be the moduli space of "Global Buildings", which is just $\mathbb{T}_G/\mathbb{K}_G$. Then there is a map

$$\mathbb{C}[\mathrm{LocSys}_S] \to \mathrm{End}\mathbb{C}[\mathfrak{B}],$$

which is compatible with Ginzburg-Kapranov's conjecture for the map $\mathcal{M}^\circ \to \mathcal{M} \to \mathfrak{B}$.

Thanks for Listening !