Derived Level Structures

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Abstract

We study the representability of relative Cartier divisor in the context of spectral algebraic geometry. Base on this, we define the derived level structures in spectral algebraic geometry. We prove the relative representability of derived level structures. Combining derived level structures and derived deformations developed by Lurie, we construct the non-even periodic higher categorical lifts of Lubin-Tate towers.

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1 Introduction

The stable homotopy category is a central topic in algebraic topology. Structured ring spectra are the most common examples studied, such as H_∞ spectra and E_∞ spectra. In

[Lur09b] and [Lur18b], Lurie uses spectral algebraic methods give a proof of the Goerss-Hopkins-Miller theorem for topological modular forms. Except for the application of elliptic cohomology, Lurie also proved the E_∞ structures of Morava E-theories [Lur18b], which use the spectral version of deformation theory of certain p-divisible groups. The earliest proof of E_∞ structures of Morava E-theories is due to Goerss, Hopkins and Miller [GH04]. They turned the problem into a moduli problem and developed an obstruction theory. One can finish the proof by computing the Andre-Quillen groups. Comparing with their method, Lurie's proof is more conceptual. There are more and more applications of spectral algebraic geometry in algebraic topology. Such as topological automorphic forms [BL10], Morava E-theories over any F_p-algebra [Lur18b], not only just for a perfect field k. The construction of equivariant topological modular forms [GM20], elliptic Hochschild homology [ST23] and more.

On the other hand, moduli problems concerning deformations of formal groups with level structures are also representable, and moduli spaces of different levels form a Lubin-Tate tower [RZ96, FGL08]. We know that the universal objects of deformations of formal groups have higher algebraic analogues which are the Morava E-theories. A natural question is what are higher categorical analogues of moduli problems of deformations with level structures? And can we find higher categorical analogues of Lubin-Tate towers. Although the $\mathbb{E}_\infty$-structure of topological modular forms with level structures can be obtained from [HL16], we still hope that there exists a derived stack of spectral elliptic curves with level structures which provide us with a more moduli interpretation. Except this, in the computation of unstable homotopy groups of sphere, after applying the EHP spectral sequences and the Bousfield-Kuhn functor, we observe that some terms on the E_2-page also arise from the universal deformation of isogenies of formal groups. They are computed by the Morava E-theories on the classifying spaces of symmetric groups [Str97, Str98]. They can be viewed as sheaves on the Lubin-Tate tower. We hope to provide a more conceptual perspective on this fact within the higher categorical Lubin-Tate tower.

In this paper, we give an attempt to address this problem by studying specific moduli problems in spectral algebraic geometry. The main ingredient of our work is the derived version of Artin's representability theorem established in [Lur04, TTV08]. We will use the spectral algebraic geometry version [Lur18c] in this paper. We study relative Cartier divisors in the context of spectral algebraic geometry. By imposing certain conditions, we define derived level structures of certain geometric objects in spectral algebraic geometry. Using Artin representability theorem, we prove some representable results of moduli problems that arise from our derived level structures. We give some examples of applications involving derived level structures. We consider the moduli problem of spectral deformations with derived level structures of p-divisible groups. We prove that these moduli problems are representable by certain formal affine spectral Deligne-Mumford stacks and the corresponding spectra can provide us many interesting general cohomology theories.

Outline

We work on spectral algebraic geometry in this paper. In the second section, we define derived isogenies and prove that the kernel of a derived isogeny in some cases have the same phenomenon as in the classical case. This provides evidence that our derived versions of level structures must induce classical level structures. For representability reasons, we use moduli associated with sheaves to detect higher homotopy of derived versions of level structures. We define relative Cartier divisors in the context of spectral algebraic geometry. For a spectral Deligne-Mumford stack X over a spectral Deligne-Mumford stack S, a relative Cartier divisor is a morphism $D \to S$ of spectral Deligne-Mumford stacks such that $D \to X$ is a closed immersion, the ideal sheaf of D is a line bundle over X, and the morphism $D \to S$ is flat, proper and locally almost of finite presentation. We use Lurie's representability theorem prove that the relative Cartier divisor is representable in certain cases. The main part of our proof involves computing of cotangent complex. Our first main result is:

Theorem A. (Theorem 2.17) Suppose that E is a spectral algebraic space over a connective $\mathbb{E}_\infty$-ring R, such that $E \to R$ is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. Then the functor

$$\begin{aligned} \mathrm{CDiv}_{E/R} \ : \ & \mathrm{CAlg}_R^{cn} \to \mathcal{S} \\ & R' \mapsto \mathrm{CDiv}(E_{R'}/R') \end{aligned}$$

is representable by a spectral algebraic space which is locally almost of finite presentation over R.

In the third section, we define derived level structures of spectral elliptic curves. Roughly speaking, for a finite abstract abelian group A, usually equals $\mathbb{Z}/N\mathbb{Z}$, $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, a derived A-level structure of a spectral elliptic curve E over an $\mathbb{E}_\infty$-ring R is just a relative Cartier divisor $D \to E$ satisfying its restriction to the heart comes from an ordinary A-level structure. We let $\mathrm{Level}(\mathcal{A}, E/R)$ denote the space derived A-level structures of a spectral elliptic curve E/R. We prove that moduli problems associated with derived level structures are representable. Our second main result is:

Theorem B. (Theorem 3.5) Suppose that E is a spectral elliptic curve over a connective $\mathbb{E}_\infty$-ring R, then the functor

$$\begin{aligned} \mathrm{Level}_{E/R} \ : \ & \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \\ & R' \mapsto \mathrm{Level}(\mathcal{A}, E_{R'}/R') \end{aligned}$$

is representable by an affine spectral Deligne-Mumford stack which is locally almost of finite presentation over the $\mathbb{E}_\infty$-ring R.

In classical algebraic geometry, except one-dimensional group curves, we also care level structures of p-divisible groups, it comes the full sections of commutative finite flat group schemes. In section three, we consider derived level structures of spectral p-divisible groups. Let $\mathrm{Level}(k, G_R/R)$ denote the space of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structures of a height h spectral p-divisible group G/R. Out third main result is:

Theorem C. (Theorem 3.16) Suppose G is a spectral p-divisible group of height h over a connective $\mathbb{E}_\infty$-ring R. Then the functor

$$\mathrm{Level}^k_{G/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}(k, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S(k) = \mathrm{Sp\acute{e}t}\mathcal{P}^k_{G/R}$.

In the last section, we give some applications of derived level structures. We first prove that the moduli problem of spectral elliptic curves with derived A-level structures is representable by a spectral Deligne-Mumford stack. Our fourth main result is:

Theorem D. (Theorem 4.7) Let $\mathrm{Ell}(\mathcal{A})(R)$ denote the space of spectral elliptic curves with derived A-level structures over the $\mathbb{E}_\infty$-ring R. The functor

$$\begin{aligned} \mathcal{M}_{ell}(\mathcal{A}) \quad : \quad & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{ell}(\mathcal{A})(R) = \mathrm{Ell}(\mathcal{A})(R) \end{aligned}$$

is representable by a spectral Deligne-Mumford stack and moreover this stack is locally almost of finite presentation over the sphere spectrum $\mathbb{S}$.

In [Lur18b], Lurie consider the spectral deformations of classical p-divisible groups. As we have the concept of derived level structures, it is natural to consider the moduli of spectral deformations with derived level structures of certain p-divisible groups. Suppose G_0 is a p-divisible group of height h over a perfect F_p-algebra R_0. We consider the following functor

$$\begin{aligned} \mathcal{M}^{or}_k \quad : \quad & \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{or}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, k)$ is the ∞-category spanned by those quaternions (G, ρ, e, η)

1. G is a spectral p-divisible group over R.
2. ρ is a equivalence class of G_0-taggings of R.
3. e is an orientation of the identity component of G.

4. $\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G/R.

Our last main result is:

Theorem E. (Theorem 4.9) The functor $\mathcal{M}_k^{or}$ is corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JL}_k$, where $\mathcal{JL}_k$ is a finite $R_{G_0}^{or}$-algebra, $R_{G_0}^{or}$ is the orientation deformation ring of G_0 defined in [Lur18b].

We will give another example of spectra constructed by considering moduli of spectral deformations with p power order subgroups level structures, which can be viewed as topological realizations of universal objects of Strickland's deformations of Frobenius.

Notations

1. CAlg denote the ∞-category of E_∞-rings, and CAlg^{cn} denote the ∞-category of connective E_∞-rings.

2. $\mathcal{S}$ denote the ∞-category of spaces (∞-groupoids).

3. For a spectral Deligne-Mumford stack $X = (\mathcal{X}, \mathcal{O}_\mathcal{X})$, we let $\tau_{\leq n}X = (\mathcal{X}, \tau_{\leq n}\mathcal{O}_\mathcal{X})$ denote its n-truncation.

4. For a spectral Deligne-Mumford stack $X = (\mathcal{X}, \mathcal{O}_\mathcal{X})$, we let $X^\heartsuit = (\mathcal{X}^\heartsuit, \tau_{\leq 0}\mathcal{O}_\mathcal{X})$ denote its underlying ordinary Deligne-Mumford stack.

5. By a spectral Deligne-Mumford stack X over R, we mean a map of spectral Deligne-Mumford stacks $X \to \mathrm{Sp\acute{e}t}R$.

6. X be a spectral Deligne-Mumford stack over R, let S be an R-algebra. We some times let $X \times_R S$ denote the product $X \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}S$.

7. $\mathcal{M}_{ell}$ denote the spectral Deligne-Mumford stack of spectral elliptic curves, which is defined in [Lur18a].

8. $\mathcal{M}_{ell}^{cl}$ denote the classical Deligne-Mumford stack of classical elliptic curves.

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2 Relative Cartier Divisors

2.1 Isogenies of Spectral Elliptic Curves

Our main innovation is derived level structures defined in this chapter. The start is derived version of isogenies. We prove that the kernel of a derived isogeny in some cases have the same phenomenon as the classical case. This gives us an evidence that over derived version of level structures must induce classical level structures. In section 2, we define relative Cartier divisors in the setting of spectral algebraic geometry. We then use Lurie's representability theorem prove that functors associated with relative Cartier divisors are representable by certain spectral Deligne-Mumford stacks. In the third and fourth section, we study derived level structures of spectral elliptic curves and spectral p-divisible groups. The main content of last two sections are the proof of representability of derived level structures.

To define derived level structures, the first question is what the higher categorical analogue of finite abelian groups are? We first recall some finiteness conditions in $\mathbb{E}_\infty$-rings context.

Let A be an $\mathbb{E}_\infty$-ring, M be an A-module. We say M is

1. perfect, if it is an compact object of $L\mathrm{Mod}_R$.
2. almost perfect, if there exits a integer k such that $M \in (L\mathrm{Mod}_R)_{\geq k}$ and M is an almost perfect object of $(L\mathrm{Mod}_R)_{\geq k}$.
3. perfect to order n if for every filtered diagram $\{N_\alpha\}$ in $(L\mathrm{Mod}_A)_{\leq 0}$, the canonical map $\varinjlim_{\alpha}\mathrm{Ext}^i_A(M, N_\alpha) \to \mathrm{Ext}^i_A(M, \varinjlim_{\alpha} N_\alpha)$ is injective for $i = n$ and bijective for $i \leq n$.
4. finitely n-presented if M is n-truncated and perfect to order (n+1).
5. finite generated, if it is perfect to order 0.

And when we consider the finite condition on algebra. We say a morphism $\phi : A \to B$ of connective $\mathbb{E}_\infty$-rings is

1. finite presentation if B belongs to the smallest full subcategory of CAlg_A^{free} and is stable under finite colimits.
2. locally of finite presentation if B is a compact object of CAlg_A.
3. almost of finite presentation if A is an almost compact object of CAlg_A, that is, $\tau_{\leq n}B$ is a compact object of $\tau_{\leq n}\mathrm{CAlg}_A$ for all $n \geq 0$.

4. finite generation to order n if the following conditions holds:

   Let $\{C_\alpha\}$ be a filtered diagram of connective $\mathbb{E}_\infty$-rings over A having colimit C. Assume that each C_α is n-truncated and that each of the transition maps $\pi_n C_\alpha \to \pi_n C_\beta$ is a monomorphism. Then the canonical map

   $$\lim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$$

   is a homotopy equivalence.

5. finite type if it is of finite generation to order 0.

6. finite if B is a finitely generated as an A-module.

Proposition 2.1. *[Lur18c, Proposition 2.7.2.1, Proposition 4.1.1.3] Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings.. Then The following conditions are equivalent.*

1. *ϕ is of finite (finite type).*

2. *The commutative ring $\pi_0 B$ is finite (finite type) over $\pi_0 A$.*

Definition 2.2 [Lur18c, Definition 4.2.0.1] Let $f : X \to Y$ be a morphism of spectral Deligne-Mumford Stack. We say that f is locally of finite type, (locally of finite genreration to order n, locally almost of finite presentation, locally of finite presentation) if the following conditions is satisfied: for every commutative diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t} B & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Sp\acute{e}t} A & \longrightarrow & Y \end{array}$$

where the horizontal morphisms are étale, the $\mathbb{E}_\infty$-ring B is finite type (finite generation to order n, almost of finite presentation, locally of finite presentation) over A.

Definition 2.3 [Lur18c, Definition 5.2.0.1] Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of spectral Deligne-Mumford stacks, we say f is finite, if the following conditions hold

1. f is affine.

2. The push-forward is $f_* \mathcal{O}_X$ is perfect to order 0 as a $\mathcal{O}_Y$ module.

Remark 2.4 By the [Lur18c, Example 4.2.0.2], A morphism $f : X \to Y$ of spectral Deligne-Mumford stack is locally of finite type if the underlying map of spectral Deligne-Mumford stacks is locally of finite type in the sense of ordinary algebraic geometry.

And by [Lur18c, 5.2.0.2], A morphism of $f : X \to Y$ is finite if the underlying map $f^\heartsuit : X^\heartsuit \to Y^\heartsuit$ is finite. If X and Y are spectral algebraic spaces, then f is finite is equivalent to $f^\heartsuit$ is finite is the sense of ordinary algebraic geometry.

We recall that a morphism $f : X \to Y$ of spectral Deligne-Mumford stacks is surjective if for every field k and any map $\mathrm{Sp\acute{e}t} k \to Y$, the fiber product $\mathrm{Sp\acute{e}t} k \times_Y X$ is nonempty [Lur18c, Definition 3.5.5.5].

Definition 2.5 Assume that we have a connective $\mathbb{E}_\infty$ ring R. Let $f : X \to Y$ be a morphism of spectral abelian varieties over R, we say f is an isogeny if it is flat, finite and surjective.

Lemma 2.6. *Let $f : X \to Y$ be a morphism of spectral abelian varieties, then $f^\heartsuit : X^\heartsuit \to Y^\heartsuit$ is an isogeny in the classical sense.*

Proof. In classical abelian varieties, $f^\heartsuit$ is an isogeny means $f^\heartsuit$ is surjective and $\ker f^\heartsuit$ is finite. But it is equivalent to $f^\heartsuit$ is finite, flat and surjective [Mil86, Proposition 7.1]. And it is easy to see that $f^\heartsuit$ is finite, flat. We only need to prove that $f^\heartsuit$ is surjective.

For every morphism $|\mathrm{Spec} k| \to |Y^\heartsuit|$, this correspond to a morphism $\mathrm{Sp\acute{e}t} k \to Y^\heartsuit$, by the inclusion-truncation adjunction [Lur18c, Proposition 1.4.6.3], this corresponds to a morphism $\mathrm{Sp\acute{e}t} k \to Y$. By the definition of surjective, we get a commutative diagram

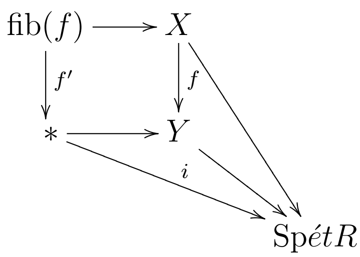
$$\begin{array}{ccc} \mathrm{Sp\acute{e}t} k' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t} k & \longrightarrow & Y \end{array}$$

The upper horizontal morphism corresponds to a morphism $\mathrm{Sp\acute{e}t} k' \to X^\heartsuit$ by inclusion-truncation adjunction. On the underlying topological space level, this corresponds to a point $|\mathrm{Sp\acute{e}t} k| \to |Y^\heartsuit|$. It is clear that this point in $|Y^\heartsuit|$ is a preimage of $|\mathrm{Sp\acute{e}t} k|$ in $X^\heartsuit$. So $f^\heartsuit$ is surjective.

■

Lemma 2.7. *Let $f : X \to Y$ be an isogeny of spectral elliptic curves over a connective $\mathbb{E}_\infty$-ring R, then* $\mathrm{fib}(f)$ *exists and is a finite and flat nonconnective spectral Deligne-Mumford stack over R.*

Proof. By [Lur18c, Proposition 1.14.1.1], the finite limits of nonconnective spectral Deligne-Mumford stacks exists, so we can define fib(f). We consider the following diagram



where the square is a pullback diagram. We find that $\mathrm{fib}(f)$ is over SpétR. By [Lur18c, Remark 2.8.2.6], $f' : \mathrm{fib}(f) \to *$ is flat because it is a pull-back of a flat morphism. Obviously $i : * \to$ SpétR is flat, so by [Lur18c, Example 2.8.3.12] (flat morphism is local on the source for the flat topology), $i \circ f' : \mathrm{fib}(f) \to$ SpétR is flat.

Next, we show $\ker f$ is finite over R. Since $*$, X and Y are all spectral algebraic spaces, so we have $\mathrm{fib} f$ is also a spectral algebraic space. And SpétR is an algebraic space [Lur18c, Example 1.6.8.2]. By the above remark 2.4, we only need to prove that the underlying morphism is finite. The truncation functor is a right adjoint , so preserve limits. So we get a pull-back diagram

$$\begin{array}{ccc} \mathrm{fib}(f)^{\heartsuit} & \longrightarrow & X^{\heartsuit} \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y^{\heartsuit} \end{array}$$

So we are reduced to prove that for an isogeny $f^{\heartsuit} : X^{\heartsuit} \to Y^{\heartsuit}$ of ordinary abelian varieties over a commutative ring R. $\ker f$ is finite over R. But this is true in classical algebraic geometry [Mil86, Proposition 7.1].

■

Lemma 2.8. *Let $f_N : E \to E$ be an isogeny of spectral elliptic curves over R, such that the underline map of ordinary elliptic curve is the multiplication N map, $N : E^{\heartsuit} \to E^{\heartsuit}$. Then* $\mathrm{fib} f$ *is finite locally free of rank N in the sense of [Lur18c, Definition 5.2.3.1]. And moreover if N is invertible in $\pi_0 R$, then* $\mathrm{fib} f$ *is a locally constant étale sheaf.*

Proof. By [KM85, Theorem 2.3.1], we know that $N : E^{\heartsuit} \to E^{\heartsuit}$ is locally free of rank N in the classical sense. When N is invertible in $\pi_0 R$, then $\ker N$ is locally constant tale sheaf. $\mathrm{fib}(f_N)$ is a spectral algebraic space which is finite and flat and its underlying map $\mathrm{fib}(f_N)^{\heartsuit} = \ker N$ is locally free of rank N. We need to prove that $\mathrm{fib} f_N \to$ SpétR is locally free of rank N in spectral algebraic geometry. But $\mathrm{fib} f_N$ is finite and flat, so is affine. We are reduce to prove this in local affine, i.e., we need ot prove that $f_N|_{\text{Spét}S}$: Spét$S \to$ SpétR is locally free, for SpétS is an affine substack of $\mathrm{fib} f_N$. This is equivalent to prove that $R \to S$ is locally free of rank N in the sense of [Lur18c, Definition 2.9.2.1]. So we need to prove

1. S is locally free of finite rank over R.(By [Lur17, Proposition 7.2.4.20], this is equivalent to say S is a flat and almost perfect R-module.)

2. For every $\mathbb{E}_\infty$-ring maps $R \to k$, the vector space $\pi_0(M \otimes_R k)$ is a N-dimensional k-vector space.

For (1), we know that $\pi_0 S$ is projective $\pi_0 R$-module, and S is a flat R-module, so by [Lur09a, Proposition 7.2.2.18], S is a projective R-module. And since $\pi_0 S$ is a finitely generate R-module, so by [Lur17, Corollary 7.2.2.9], S is a retract of a finitely generated free R-module M, so is locally free of finite rank.

For (2), $\pi_0(k \otimes_R M)$ since R and M are connective, by [Lur17, Corollary 7.2.1.23], we get $\pi_0(k \otimes_R M) \simeq k \otimes_{\pi_0 R} \pi_0 M$ is a rank N k-vector space ($\pi_0 M$ is rank N free $\pi_0 R$ module).

We next show that if N is invertible in $\pi_0 R$, then $\mathrm{fib} f$ is a locally constant sheaf. By the above discussion, $\mathrm{fib} f$ is a spectral Deligne-Mumford stack, so the associated functor points $\mathrm{fib} f : \mathrm{CAlg}_R \to S$ is nilcomplete and locally of almost finite presentation. By [KM85, Theorem 2.3.1], $\mathrm{fib} f|_{\mathrm{CAlg}^{\heartsuit}_{\pi_0 R}}$ is a locally constant sheaf, the desired results follows form the following lemma.

■

Lemma 2.9. *Let $\mathcal{F} \in \mathrm{Shv}^{\acute{e}t}(\mathrm{CAlg}^{\mathrm{cn}}_R)$, and is nilcomplete, locally of almost finite presentation and $\mathcal{F}|_{(\mathrm{CAlg}^{\mathrm{cn}}_R)^{\heartsuit}}$ is the associated sheaf of constant presheaf valued on A. Then $\mathcal{F}$ is a homotopy locally constant sheaf (i.e., sheafification of a homotopy constant presheaf).*

Proof. We choose a tale cover U_i^0 of $\pi_0 R$, such that $\mathcal{F}|_{U_i^0}$ is a constant sheaf for each i. By [Lur17, Theorem 7.5.1.11], this corresponds to an tale cover $U_i \to R$ such that $\pi_0 U_i = U_i^0$. We consider the following diagram

$$\begin{array}{ccc} \tau_{\leq 0} R & \longrightarrow & \tau_{\leq 0} U \\ \downarrow & & \downarrow \\ \tau_{\leq n} R & \longrightarrow & \tau_{\leq n} U \end{array}$$

which is push-out diagram, since U_i is an étale R algebra. This is a colimit diagram in $\tau_{\leq n}\mathrm{CAlg}_R$. $\mathcal{F}$ is a sheaf of locally of almost finite prsentation, so we get push-out diagram

$$\begin{array}{ccc} \mathcal{F}(\tau_{\leq 0} R) & \longrightarrow & \mathcal{F}(\tau_{\leq 0} U_i) \\ \downarrow & & \downarrow \\ \mathcal{F}(\tau_{\leq n} R) & \longrightarrow & \mathcal{F}(\tau_{\leq n} U_i) \end{array}$$

For each i, we have such diagram. Without loss of generality, we can assume each U_i is connective. So $\mathcal{F}(\tau_{\leq 0} U_i)$ are always same for all i. That means we have $\mathcal{F}(\tau_{\leq n} U_i)$ are all equivalence. But we have $\mathcal{F}$ is nicomplete, this means $\mathcal{F}(U_i) \simeq \mathrm{colim} \mathcal{F}(\tau_{\leq n} U_i)$. So we get all $\mathcal{F}(U_i)$ are homotopy equivalence.

■

2.2 Relative Cartier Divisors

In the subsection, we will define the relative Cartier divisor in the context of Spectral Algebraic Geometry. And we use Lurie's spectral Artin's representability theorem to prove that relative Cartier divisor is representable in some good cases. We first recall the following spectral Artin's representability theorem.

Theorem 2.10. *[Lur18c, Theroem 18.3.0.1] Let $X : \mathrm{CAlg}^{cn} \to \mathcal{S}$ be a functor, if we have a natural transformation $f : X \to \mathrm{Spec}R$, where R is a Noetherian $\mathbb{E}_\infty$-ring and $\pi_0 R$ is a Grothendieck ring. For $n \geq 0$, X is representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over R if and only if the following conditions are satisfied:*

1. *For every discrete commutative ring R_0, the space $X(R_0)$ is n-truncated.*
2. *The functor X is a sheaf for the étale topology.*
3. *The functor X is nilcomplete, infinitesimally cohesive, and integrable.*
4. *The functor X admits a connective cotangent complex L_X.*
5. *The natural transformation f is locally almost of finite presentation.*

For a locally spectrally topoi $X = (\mathcal{X}, \mathcal{O}_x)$, we can consider its functor of points

$$h_X : \infty\mathrm{Top}^{loc}_{\mathrm{CAlg}} \to \mathcal{S}, \quad Y \mapsto \mathrm{Map}_{\infty\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}}(Y, X)$$

By [Lur18c, Remark 3.1.1.2], the closed immersion of locally spectrally ringed topos $f : X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to Y = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ corresponds to morphism of sheaves of connective $\mathbb{E}_\infty$-rings $\mathcal{O}_{\mathcal{X}} \to f_*\mathcal{O}_{\mathcal{Y}}$ over $\mathcal{X}$ such that $\pi_0\mathcal{O}_{\mathcal{X}} \to \pi_0 f_*\mathcal{O}_{\mathcal{Y}}$ is surjective. We consider the the fiber of this map $\mathrm{fib}f$. For a closed immersion $f : D \to X$ of spectral Deligne-Mumford stack, we let $I(D)$ denote $\mathrm{fib}(f)$, called the ideal sheaf of D.

To prove the relative representability, we need the representability of the Picard functor. If we have a map $f : X \to \mathrm{Sp\acute{e}t}R$ of spectral Deligne-Mumford stack, we can define a functor

$$\mathscr{P}ic_{X/R} : \mathrm{CAlg}^{cn}_R \to \mathcal{S}, \quad R' \mapsto \mathscr{P}ic(\mathrm{Sp\acute{e}t}R' \times_{\mathrm{Sp\acute{e}t}R} X)$$

If f admits a section $x : \mathrm{Sp\acute{e}t}R \to X$ then there exists a natural transformation of functors $\mathscr{P}ic(X/R) \to \mathscr{P}ic_{R/R}$. We let

$$\mathscr{P}ic^x_{X/R} : \mathrm{CAlg}^{cn}_R \to \mathcal{S}$$

denote the fiber of this map.

Theorem 2.11. *[Lur18c, Theorem 19.2.0.5] Let X be a map spectral algebraic spaces which is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected over an $\mathbb{E}_\infty$-ring R. And suppose that $x : \text{Sp\'et}R \to X$ is a section, the functor $\mathscr{P}ic^x_{X/R}$ is representable by a spectral algebraic space which is locally of finite presentation over R.*

In the classical case, relative Cartier divisors schemes are open subschemes of Hilbert schemes [Kol13]. But in the derived case, the Hilbert functor is representable by a spectral algebraic space [Lur04, Theorem 8.3.3], it is hard to say relation to say the relation between them. We will directly study relative Cartier divisors in derived world.

Definition 2.12 Suppose that X is a spectral Deligne-Mumford stack over a spectral Deligne-Mumford stack S. We let $\mathrm{CDiv}(X/S)$ denote the ∞-category of closed immersions $D \to X$, such that D is flat, proper, locally almost of finite presentation over S and the associated ideal sheaf of D is locally free of rank one over X.

Remark 2.13 It is easy to say that for any spectral Deligne-Mumford stack X over S, $\mathrm{CDiv}(X/S)$ is a kan complex, since all objects are closed immersions of X, let $D \to D'$ be morphism, then we have a diagram

$$\begin{array}{ccc} D & \xrightarrow{\quad f \quad} & D' \\ & \searrow \qquad \swarrow & \\ & X & \end{array}$$

by the definition of closed immersions, they all equivalent to the same substack of X, so f is a equivalence.

Lemma 2.14. *Let X/S be a spectral Deligne-Mumford stack, and $T \to S$ be a map of spectral Deligne-Mumford stacks. If we have a relative Cartier divisor $i : D \to X$, then D_T is a relative Cartier divisor of X_T.*

Proof. This is easy to see, we just notice that D_T is still closed immersion of X_T [Lur18c, Corollary 3.1.2.3]. And after base change, D_T is flat, proper, locally almost of finite presentation over T. The only thing we need to worry is that whether $I(D_T)$ is a line bundle over X_T? But this is also true. Since we have a fiber sequence

$$I(D) \to \mathcal{O}_X \to \mathcal{O}_D$$

after applying the morphism $f^* : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_{X_T}}$, due to the flatness of D. We get fiber sequence

$$f^*(I(D)) \to \mathcal{O}_{X_T} \to \mathcal{O}_{D_T}$$

So we get $I(D_T)$ is just $f^*I(D)$, so is invertible.

■

By the construction of relative Cartier divisors, suppose that X is a spectral Deligne-Mumford stack over an affine spectral Deligne-Mumford stack $S = \mathrm{Sp\acute{e}t} R$. We then have a functor

$$\begin{aligned} \mathrm{CDiv}_{X/R} \quad : \quad & \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \\ & R' \mapsto \mathrm{CDiv}(E_{R'}/R') \end{aligned}$$

Our main target in this section is to prove this functor is representable when E/R is a spectral algebraic space satisfying certain conditions. Before we start the prove of represenability of relative Cartier divisor, we need some preparations for computing the cotangent complex of a relative Cartier divisor functor. The main issiue is square extension. We need following truth about pushout of two closed immersions.

By [Lur18c, Theorem 16.2.0.1, Proposition 16.2.3.1], suppose we have a pushout square of spectral Deligne-Mumford stacks:

$$\begin{array}{ccc} X_{01} & \xrightarrow{i} & X_0 \\ \downarrow j & & \downarrow j' \\ X_1 & \xrightarrow{i'} & X, \end{array}$$

such that i and j are closed immersions. Then the induced square of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(X_{01}) & \longleftarrow & \mathrm{QCoh}(X_0) \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(X_1) & \longleftarrow & \mathrm{QCoh}(X) \end{array}$$

determines emdbedding $\theta : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X_0) \times_{\mathrm{QCoh}(X_{01})} \mathrm{QCoh}(X_1)$ and restricts to an equivalence

$$\mathrm{QCoh}(X)^{\mathrm{cn}} \to \mathrm{QCoh}(X_0)^{\mathrm{cn}} \times_{\mathrm{QCoh}(X_{01})^{\mathrm{cn}}} \mathrm{QCoh}(X_1)^{\mathrm{cn}}$$

Let $\mathcal{F} \in \mathrm{QCoh}(X)$, and set

$$\mathcal{F}_0 = j'^{*} \in \mathrm{QCoh}(X_0) \quad \mathcal{F}_1 = i'^{*}\mathcal{F} \in \mathrm{QCoh}(X_1).$$

Then the quasi-coherent sheaf $\mathcal{F}$ is n-connective is equivalent $\mathcal{F}_0$ and $\mathcal{F}_1$ are n-connective, and this statement is also true for the condition, almost connective, Tor-amplitude $\leq n$ flat, perfect to order n, almost perfec, perfect, locally free of finite rank.

And by [Lur18c, Theorem 16.3.0.1], we the have a pullback square

$$\begin{array}{ccc} \mathrm{SpDM}_{/X} & \longrightarrow & \mathrm{SpDM}_{/X_0} \\ \downarrow & & \downarrow \\ \mathrm{SpDM}_{/X_1} & \longrightarrow & \mathrm{SpDM}_{/X_{01}} \end{array}$$

of ∞-categories Let $f : Y \to X$ be a map of spectral Deligne-Mumford stacks. Let $Y_0 = X_0 \times_X Y$, $Y_1 = X_1 \times_X Y$ and let $f_0 : Y_0 \to X_0$ and $f_1 : Y_1 \to X_1$ be the projections maps. Then we have [Lur18c, Proposition 16.3.2.1] f is locally almost of finite presentation if and only if both f_0 and f_1 are locally almost of finite presentation. And the statement is also trur for conditions: locally of finite generation to order n, locally of finite presentation, étale, equivalence, open immersion, closed immersion, flat, affine, separated and proper.

Let $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack, and $\mathcal{E} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$ is a quasi-coherent sheaf, and $\eta \in \mathrm{Der}(\mathcal{O}_X, \Sigma\mathcal{E})$, that is map $\eta : \mathcal{O}_X \to \mathcal{O}_X \oplus \Sigma\mathcal{E}$. We let $\mathcal{O}_X^{\eta}$ denote the square-zero extension of $\mathcal{O}_X$ by $\mathcal{E}$ determined by η, then we have a pull-back diagram

$$\begin{array}{ccc} \mathcal{O}_X^{\eta} & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \eta \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_X \oplus \Sigma\mathcal{E} \end{array}$$

By [Lur18c, Proposition 17.1.3.4], $(\mathcal{X}, \mathcal{O}_X^{\eta})$ is a spectral Deligne-Mumford stack, which we will denote it by $\mathcal{X}^{\eta}$. In the case of $\eta = 0$, we denote it by $X^{\mathcal{E}} = (\mathcal{X}, \mathcal{O}_X \oplus \mathcal{E})$. We then have a pullback square of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} X^{\Sigma\mathcal{E}} & \xrightarrow{g} & X \\ \downarrow f & & \downarrow \\ X & \longrightarrow & X^{\eta} \end{array}$$

such that f and g are closed immersions.

We have a pullback diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X^{\eta})^{\mathrm{acn}} & \longrightarrow & \mathrm{QCoh}(X)^{\mathrm{acn}} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(X)^{\mathrm{acn}} & \longrightarrow & \mathrm{QCoh}(X^{\Sigma\mathcal{E}})^{\mathrm{acn}}. \end{array}$$

by [Lur18c, Theorem 16.2.0.1, Proposition 16.2.3.1]. Taking $\eta = 0$ and passing ti homotopy fiber over some $\mathcal{F} \in \mathrm{QCoh}(X)^{\mathrm{acn}}$, we can get

$$\mathrm{QCoh}(X^{\mathcal{E}})^{\mathrm{acn}} \times_{\mathrm{QCoh}(X)} \{\mathcal{F}\} \simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{F}, \Sigma(\mathcal{E} \otimes \mathcal{F}))$$

by [Lur18c, Proposition 19.2.2.2].

Taking $\eta = 0$ and passing to the homotopy fibers over some $Z \in \mathrm{SpDM}_{/X}$, we can get classification of the first order deformations

$$\mathrm{SpDM}_{/X^{\mathcal{E}}} \times_{\mathrm{SpDM}_{/X}} \{Z\} \simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(L_{Z/X}, \Sigma f^* \mathcal{E}),$$

see details in [Lur18c, Porposition 19.4.3.1].

Lemma 2.15. *Let $f : X \to \mathrm{Sp\acute{e}t}R$ be a morphism of spectral Deligne-Mumford stacks. For a connective R-module M, then the ∞-categories of Deigne-Mumford stacks X' with a morphism $X \to \mathrm{Sp\acute{e}t}(R \oplus M)$ such that fitting into the following pull back diagram*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}R & \longrightarrow & \mathrm{Sp\acute{e}t}R \oplus M \end{array}$$

is a Kan complex, which is canonically equivalent to the mapping space $\mathrm{Map}_{\mathrm{QCoh}}(L_{X/Y}, \Sigma f^* M)$, and moreover if f is flat, proper and locally of almost finite presnetation, then any such $f' : X' \to S[M]$ is flat, proper and locally almost of finite presentation.

Proof. We have a pullback square in $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R \oplus M & \longrightarrow & R \\ \downarrow & & \downarrow {\scriptstyle (Id,0)} \\ R & \longrightarrow & R \oplus \Sigma M, \end{array}$$

this corresponds a pushout square of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}R \oplus \Sigma M & \longrightarrow & \mathrm{Sp\acute{e}t}R \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}R & \longrightarrow & \mathrm{Sp\acute{e}t}R \oplus M \end{array}$$

such that $\mathrm{Sp\acute{e}t}R \oplus \Sigma M \to \mathrm{Sp\acute{e}t}R$ are closed immersion. That makes $\mathrm{Sp\acute{e}t}R \oplus M$ be an infinitesimal thickening of $\mathrm{Sp\acute{e}t}R$ determined by $R \xrightarrow{(id,0)} R \oplus \Sigma M$.

The first part of this lemma is just the formula of first order deformations[Lur18c, Proposition 19.4.3.1], and the second part is properties of pushout of two closed immersions [Lur18c, Corollary 19.4.3.3].

■

Lemma 2.16. *Suppose that we are given a pushout diagram of spectral Deligne-Mumford stacks σ:*

$$\begin{array}{ccc} X_{01} & \xrightarrow{i} & X_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ X_1 & \longrightarrow & X, \end{array}$$

where i and j are closed immersions. Let $f : Y \to X$ be a map of spectral Deligne-Mumford stacks. Let $Y_0 = X_0 \times_X Y$, $Y_1 = X_1 \times_X Y$ and let $f_0 : Y_0 \to X_0$ and $f_1 : Y_1 \to X_1$ be the projections maps.

If both f_0 and f_1 are closed immersions and determine line bundles over Y_0 and Y_1, then f is a closed immersion and determines a line bundle.

Proof. The closed immersion part is just Lurie's theorem. And for the line bundle part, we notice that by [Lur18c, Theorem 16.2.0.1, Proposition 16.2.3.1], f determine a sheaf of locally free of finite rank. To prove it is a line bundle, we can do it locally. By [Lur18c, Theorem 16.2.0.2], for a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01} \end{array}$$

of E_∞-rings such that $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective, then there is an equivalence $F : \mathrm{Mod}_A^{cn} \to \mathrm{Mod}_{A_0}^{cn} \times_{\mathrm{Mod}_{A_{01}}^{cn}} \mathrm{Mod}_{A_1}^{cn}$. Actually this a symmetric monoidal equivalence. Sice we have $F(M) = (A_0 \otimes_A M, A_{01} \otimes_A M, A_1 \otimes_A M)$. They satisfying $F(M \otimes N) \simeq F(M) \otimes F(N)$. But by [Lur18c, Propsition 2.9.4.2], line bundles of $A_1, A_{0,1}$ and A_0 determines invertible objects of $\mathrm{Mod}_{A_1}^{cn}, \mathrm{Mod}_{A_{01}}^{cn}$ and $\mathrm{Mod}_{A_1}^{cn}$, so determine a invertible object of Mod_A^{cn}, hence a line bundle over A by [Lur18c, Propsition 2.9.4.2].

■

Theorem 2.17. *Let E/R be a spectral algebraic space which is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. Then the functor*

$$\begin{aligned} \mathrm{CDiv}_{E/R} \ : \ & \mathrm{CAlg}_R \to \mathcal{S} \\ & R' \mapsto \mathrm{CDiv}(E_{R'}/R') \end{aligned}$$

is representable by a spectral algebraic space which is locally almost of finite presentation of R.

Proof. We use Lurie's spectral Artin's represnetability theorem to prove this theorem.

1. For every discrete commutative R_0, the space $\mathrm{CDiv}_{E/R}(R_0)$ is 0-truncated.

   We just notice that $\mathrm{CDiv}_{E/R}(R_0)$, consists of closed immersions $D \to E \times_R R_0$, such that D is flat proper over R_0, so all D are discrete object, so $\mathrm{CDiv}_{E/R}(R_0)$ is 1-truncated.

2. The functor $\mathrm{CDiv}_{E/R}$ is a sheaf for the tale topology.

   Let $\{R' \to U_i\}_{i \in I}$ be an étale cover of R', and $U_\bullet$ be the associate check simplicial object. We need to prove that the map

   $$\mathrm{CDiv}_{E/R}(R') \to \lim_{\Delta} \mathrm{CDiv}_{E/R}(U_\bullet)$$

   is an equivalence. Unwinding the definitions, we only need to prove following general result: for a spectral Deligne-Mumford stack $X \to S$ and we have a étale cover $T_i \to S$, then

   $$\mathrm{CDiv}(X/S) \to \lim_{\Delta} \mathrm{CDiv}(X \times_S T_\bullet)$$

   is a homotopy equivalence. But this obvious, since our conditions of relative Cartier divisor is local for the étale topology.

3. The functor $\mathrm{CDiv}_{E/R}$ is nilcomplete.

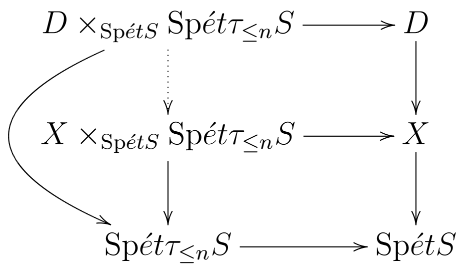
   This is equivalent to say that the canonical map

   $$\mathrm{CDiv}_{E/R}(R') \to \varprojlim \mathrm{CDiv}_{E/R}(\tau_{\leq n} R')$$

   This can be deduced form the following results: for a flat, proper, locally almost of finite presentation spectral spectral algebraic space X over a connective E_∞-ring S, we have a equivalence

   $$\mathrm{CDiv}(X/\mathrm{Sp\acute{e}t}S) \to \varprojlim \mathrm{CDiv}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n} S).$$

   Let us prove this equivalence now. For a relative Cartier divisor $D \to X$, we have the following commutative diagram



We then get a induce map $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S \to X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$. It is easy to prove that this map is a closed immersion [Lur18c, Corollary 3.1.2.3], and $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S \to \mathrm{Sp\acute{e}t}S$ is flat, proper and locally almost of finite presentation, since $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ is the base change of D along $\mathrm{Sp\acute{e}t}\tau_{\leq n}S \to \mathrm{Sp\acute{e}t}S$, and the associated ideal sheaf of $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ is still a line bundle over $X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$. So $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ is a relative Cartier divisor of $X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$. Thus we have define a functor

$$\theta : \mathrm{CDiv}(X/S) \to \varprojlim \mathrm{CDiv}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S), \quad D \mapsto \{D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S\}$$

This functor is fully faithful, since we have equivalence $\mathrm{SpDM}_{/S} \to \varprojlim \mathrm{SpDM}_{/\tau_{\leq n}S}$ defined by $X \mapsto X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ [Lur18c, Proposition 19.4.1.2]. To prove the functor θ is an equivalence, we need to show it is essentially surjective. Suppose $\{D_n\} \to X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ is an object in $\varprojlim \mathrm{CDiv}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S)$. It is a morphism in $\varprojlim \mathrm{SpDM}_{/\tau_{\leq n}S}$, by [Lur18c, Proposition 19.4.1.2], there is a morphism $D \to X$ in $\mathrm{SpDM}_{/S}$, satisfying $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S \to X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ are just $D_n \to X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$.

Next, we need to show that such $D \to X$ is relative Cartier divisor. The condition that $D \to S$ is flat, proper and locally almost of finite presentation follows immediately from [Lur18c, Proposition 19.4.2.1]. We need to prove that $D \to X$ is a closed immersion and determine a line bundle over X. Without loss of generality, we may assume that $X = \mathrm{Sp\acute{e}t}B$ is affine, so we have closed immersion $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S \to \mathrm{Sp\acute{e}t}B \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S \simeq \mathrm{Sp\acute{e}t}(B \otimes_S \tau_{\leq n}S)$, the second equivalence comes from [Lur18c, Proposition 1.4.11.1(3)]. And by [Lur18c, Theorem 3.1.2.1], $D \times_{SpetS} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$ equals $\mathrm{Sp\acute{e}t}B'_n$ for each n, such that $\pi_0(B \times_S \tau_{\leq n}S) \to \pi_0 B'_n$ is surjective. Since we have $\tau_{\leq n}S \to B'_n$ is flat, we get $\mathrm{Sp\acute{e}t}B'_n = \mathrm{Sp\acute{e}t}B'_{n+1} \times_{\mathrm{Sp\acute{e}t}\tau_{\leq n+1}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S = \mathrm{Sp\acute{e}t}(B'_{n+1} \times_{\tau_{\leq n+1}S} \tau_{\leq n}S) \simeq \mathrm{Sp\acute{e}t}\tau_{\leq n}B'_{n+1}$. So we get a spectrum B' such that $\tau_{\leq n}B' \simeq \mathrm{Sp\acute{e}t}B'_n = D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Sp\acute{e}t}\tau_{\leq n}S$. Consequently $D = \mathrm{Sp\acute{e}t}B'$, and $\pi_0 B \to \pi_0 B'$ is surjective, so $D = \mathrm{Sp\acute{e}t}B' \to \mathrm{Sp\acute{e}t}B = X$ is a closed immersion. To prove that the associated ideal sheaf of D is a line bundle, we notice that there is a pullback diagram.

$$\begin{array}{ccc} I_n & \longrightarrow & B \times_S \tau_{\leq n}S \\ \downarrow & & \downarrow \\ * & \longrightarrow & B' \times_S \tau_{\leq n}S, \end{array}$$

each I_n is an invertible $B \times_S \tau_{\leq n}S = \tau_{\leq n}B$ module. Passing to the inverse limit, we get

$$\begin{array}{ccc} \varprojlim I_n & \longrightarrow & B \\ \downarrow & & \downarrow \\ * & \longrightarrow & B'. \end{array}$$

Consequently, we have $I(D) \simeq \varprojlim I_n$. So by the nilcompleteness of Picard functor [Lur18c, Corollary 19.2.4.6, Propostion 19.2.4.7], We get I is a invertible B-module. So the associated ideal sheaf of D is a line bundle of X.

4. The functor $\mathrm{CDiv}_{E/R}$ is cohesive.

   This statement follows from Proposition 2.16 and [Lur18c, Proposition 16.3.2.1].

5. The functor $\mathrm{CDiv}_{E/R}$ is integrable. We need to prove that for R' a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal $m \subset \pi_0 R$. Then the inclusion functor induces a homotopy equivalence

$$\mathrm{Map}_{Fun(\mathrm{CAlg}^{cn},\mathcal{S})}(\mathrm{Sp\acute{e}t}R', \mathrm{CDiv}_{E/R}) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{cn},\mathcal{S})}(\mathrm{Spf}R', \mathrm{CDiv}_{E/R}).$$

   But this follows from the following result: for a flat proper, locally almost of finite presentation and separated spectral spectral algebraic space X over a connective E_∞-ring S, we have equivalence

$$\mathrm{CDiv}(X/S) \simeq \mathrm{CDiv}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S)$$

   Let $\mathrm{Hilb}(X/S)$ denote the full subcategory of $\mathrm{SpDM}_{/X}$ consists of those $D \to X$, such that $D \to X$ is a closed immersion and $D \to S$ is flat, proper and locally almost of finite presentation. Then by the formal GAGA theorem [Lur18c, Theorem 8.5.3.4] and base change properties of flat, proper and locally almost of finite presentation, we have $\mathrm{Hilb}(X/S) \simeq \mathrm{Hilb}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S)$. To prove the equivalence of relative Cartier divisors, we need to check that $D \to X$ associated a line bundle over X if and only if $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S$ associated a line bundle over $X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S$. We notice that since $X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S$ is flat over X, we have $I(D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S) = I(f^*D) \simeq f^*I(D)$

$$\begin{array}{ccc} D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S & \longrightarrow & D \\ \downarrow & & \downarrow \\ X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S & \xrightarrow{f} & X. \end{array}$$

   By [Lur18c, Proposition 19.2.4.7], we have an equivalence

$$\mathrm{QCoh}(X/S)^{\mathrm{aperf,cn}} \simeq \mathrm{QCoh}(X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S)^{\mathrm{aperf,cn}}$$

By restricting to subcategories spanned by invertible object and using [Lur18c, Proposition 2.9.4.2], we get D associated a line bundle over X if and only if $D \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S$ associated a line bundle over $X \times_{\mathrm{Sp\acute{e}t}S} \mathrm{Spf}S$.

6. $\mathrm{CDiv}_{E/R}$ is locally almost of finite presentation.

   We need to prove that $\mathrm{CDiv}_{E/R} : \mathrm{CAlg}_R \to \mathcal{S}, R' \mapsto \mathrm{CDiv}(E_{R'}/R')$ commutate with filtered colimits when restrict to $\tau_{\leq n}\mathrm{CAlg}_R^{\mathrm{cn}}$. But we notice that $\mathrm{CDiv}(E_{R'}/R')$ are full categories of $\mathrm{SpDM}_{/E_{R'} \to R'}$, we consider the functor

   $$R' \mapsto \mathrm{Var}^+_{/E_{R'} \to R}$$

   where $\mathrm{Var}^+_{/E_{R'} \to R}$ consists of the diagram

   $$\begin{array}{ccc} D & \longrightarrow & E_{R'} \\ & \searrow & \downarrow \\ & & \mathrm{Sp\acute{e}t}R' \end{array}$$

   such that $D \to R'$ is flat, proper, and locally almost of finite presentation. Then by [Lur18c, Proposition 19.4.2.1]. This functor commutates with filtered colimits when restrict to $\tau_{\leq n}\mathrm{CAlg}_R^{\mathrm{cn}}$. Then we just need to prove that when $\{D_i \to E^i_{R'}\}_{i \in I}$ are closed immersions and determine line bundles in $\{E^i_{R'}\}$, then $\mathrm{colim} D_i$ are closed immersion of $\mathrm{colim} E^i_{R'}$ and determine line bundle in $\mathrm{colim} E^i_{R'}$. But this fact follows from the locally almost of finite presentationnes of Picard functor and properties of closed immersions.

   Consider the functor $\mathrm{CDiv}_{E/R} \to *$, it is infitesimally cohesive and admits a cotangent complex which is almost perfect, so by [Lur18c, 17.4.2.2], it is locally almost of finite presentation. So $\mathrm{CDiv}_{E/R}$ is locally almost of finite presentation, since $*$ is a final object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.

7. The functor $\mathrm{CDiv}_{E/R}$ admits a complex L which is connective and almost perfect.

   For a connective E_∞-ring S, and every $\eta \in \mathrm{CDiv}_{E/R}(S)$, and a connective S-module M. We have a pullback diagram

   $$\begin{array}{ccc} F_\eta(M) & \longrightarrow & \mathrm{CDiv}_{E/R}(S \oplus M) \\ \downarrow & & \downarrow \\ \eta & \longrightarrow & \mathrm{CDiv}_{E/R}(S) \end{array}$$

Then we have a functor

$$F_\eta : \mathrm{Mod}_S \to \mathcal{S}, \quad M \mapsto F_\eta(M)$$

We need to prove that this functor is corepresentable. η corresponds a morphism $D \to E \times_R S$, and $E \times_R (S \oplus M)$ is a square zero extension of $E \times_R S$. So by the classification of first order deformation theory [Lur18c, Propostion 19.4.3.1], the space of D', which satisfying the pullback diagram

$$\begin{array}{ccc} D & \longrightarrow & D' \\ \downarrow{\scriptstyle f} & & \downarrow \\ E \times_R S & \longrightarrow & E \times_R (S \oplus M) \\ \downarrow{\scriptstyle p} & & \downarrow \\ \mathrm{Sp\acute{e}t} S & \longrightarrow & \mathrm{Sp\acute{e}t}(S \oplus M) \end{array}$$

is equivalent to

$$\mathrm{Map}_{\mathrm{QCoh}(D)}(L_{D/E\times_R S}, \Sigma f^*\mathcal{E}) = \mathrm{Map}_{\mathrm{QCoh}(D)}(L_{D/E\times_R S}, \Sigma f^* \circ p^* M)$$

Push forward along $p \circ f$, and by [Lur18c, Proposition 6.4.5.3] we have

$$\mathrm{Map}_{\mathrm{QCoh}(D)}(L_{D/E\times_R S}, \Sigma f^* \circ p^* M) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Sp\acute{e}t} S)}(\Sigma^{-1} p_+ \circ f_+ L_{D/E\times_{\mathrm{Sp\acute{e}t} R}\mathrm{Sp\acute{e}t} S}, M).$$

And by [Lur18c, Proposition 16.3.2.1] and Lemma 2.16, any such D' is a closed immersion of $\mathrm{CDiv}_{E/R}(S \oplus M)$ and determine a line bundle of $\mathrm{CDiv}_{E/R}(S \oplus M)$. Since the diagram

$$\begin{array}{ccc} D & \longrightarrow & D' \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t} S & \longrightarrow & \mathrm{Sp\acute{e}t} S \oplus M \end{array}$$

is a pullback diagram, so D' is a square zero extension of D. By [Lur18c, Proposition 16.3.2.1], we get $D' \to \mathrm{Sp\acute{e}t}(S \oplus M)$ is flat, proper and locally almost of finite presentation. Combining these facts, we find that

$$F_\eta(M) = Map_{\mathrm{QCoh}(\mathrm{Sp\acute{e}t} S)}(\Sigma^{-1} p_+ \circ f_+ L_{D/E\times_{\mathrm{Sp\acute{e}t} R}\mathrm{Sp\acute{e}t} S}, M).$$

Consequently, the functor $\mathrm{CDiv}_{E/R}$ satisfies condition (a) of [Lur18c, Example 17.2.4.4] and condition (b) follows form the compatibility of f_+ with base change. It then follows that $\mathrm{CDiv}_{E/R}$ admits a cotangent complex $L_{\mathrm{CDiv}_{E/R}}$ satisfying $\eta^* L_{\mathrm{CDiv}_{E/R}} =$

$\Sigma^{-1}p_+ \circ f_+ L_{D/E\times_{\text{Spét}R}\text{Spét}S}$. Since the quasi-coherent sheaf $L_{D/E\times_{\text{Spét}R}\text{Spét}S}$ is connective and almost perfect. The R-module $\Sigma^{-1}p_+ \circ f_+ L_{D/E\times_{\text{Spét}R}\text{Spét}S}$ is (-1) connective.

$L_{\text{CDiv}_{E/R}}$ is almost perfect, since we have $\text{CDiv}_{E/R}$ it is infitesimally cohesive and admits a cotangent complex. And it is locally almost of finite presentation, so by [Lur18c, 17.4.2.2], its cotangent complex is almost perfect.

We next show that it is connective. Let R' be an $\mathbb{E}_\infty$-ring, and $\eta \in \text{CDiv}(E_{R'}/R)$, we wish to prove that $M = \eta^* L_{\text{CDiv}_{E/R}} \in \text{Mod}'_R$ is connective. We already know that M is is (-1)-connective and almost perfect, the homotopy group $\pi_{-1}M$ is a finitely generated $\pi_0 R'$ module. To prove that π_{-1} vanishes. By the Nakayama's lemma, this is equivalent to prove that

$$\pi_{-1}M(k \otimes_{R'} M) \simeq \text{Tor}_0^{\pi_0 R'}(k, \pi_{-1}M)$$

equals to 0 for every residue filed of R. Then we may replace R' by k and assume k is a algebraically closed filed.

Let $A = k[t]/(t^2)$, unwinding the definitions, we find that the dual space $\text{Hom}_k(\pi_{-1}M, k)$ can be identify with the set of automorphism of η_A such that it restrict identity of η. we wish to prove this set is trivial. But this follow from the fact : Let X/k be scheme, L is an line bundle on X, if L_A is also a line bundle of X_A. If we have f is an automorphism of L_A such that $f|L$ is identity on L, then f is the identity. (This fact follows from the connectiveness of cotangent complexes of Picard functors.)

■

3 Derived Level Structures

3.1 Derived Level Structures of Spectral Elliptic Curves

Let C be a one dimensional smooth commutative group scheme over a base scheme S, and A be an abstract finite abelian group. A homomorphism of abstract groups

$$\phi : A \to C(S)$$

is said to be an A-Level structure on C/S if the effective Cartier divisor D in C/S defined by

$$D = \Sigma_{a\in A}[\phi(a)]$$

is a subgroup of C/S.

The following result due to Katz-Mazur [KM85] give the representability of level structures moduli problems.

Proposition 3.1. *[KM85, Proposition 1.6.2] Let C/S be an one dimensional smooth commutative group scheme over S. Then the functor*

$$\mathrm{Level}_{C/S} : \mathrm{Sch}_S \to \mathrm{Set}$$

$$T \mapsto \textit{the set of } A\textit{-level structures on } C_T/T$$

is representable by a closed subscheme of $\mathrm{Hom}(A, C) \cong C[N_1] \times_S \cdots \times_S C[N_r]$.

Definition 3.2 Let E/R be a spectral elliptic curve. In the level of objects, a derived A-level structure is a relative Cartier divisor $\phi : D \to E$ of E, such that the underlying morphism $D^{\heartsuit} \to E^{\heartsuit}$ is the inclusion of the associated relative Cartier divisor $\Sigma_{a \in A}[\phi_0(a)]$ into $E^{\heartsuit}$, where $\phi_0 : A \to E^{\heartsuit}(R^{\heartsuit})$ is any classical level structure. We let $\mathrm{Level}(\mathcal{A}, E/R)$ denote the ∞-category of derived A-level structures of E/R, whose objects can be viewed as pairs $\phi = (D, \phi)$.

It is easy to see that for a spectral elliptic curve E/R, the ∞-category $\mathrm{Level}(\mathcal{A}, E/R)$ is a ∞-groupoid, since it is a full subcategory of $\mathrm{CDiv}(E/R)$, which is a ∞-groupoid.

Lemma 3.3. *Let E/R be a spectral elliptic curve and $\phi_S : D \to E$ be a derived level structure. Suppose that $T \to S$ be a morphism of nonconnective spectral Deligne-Mumford stacks, then the induce morphism $\phi_S : D_T \to E_T$ is a derived level structure of E_T/T.*

Proof. We notice that derived level structure is stable under base change. So $\phi_S^{\heartsuit} : A \to (E \times_S T)^{\heartsuit}(T_0) = E^{\heartsuit}(T_0)$ is classical level structure, so $D_T^{\heartsuit}$ is the associated classical relative Cartier divisor of a classical level structure. And $D_T \to E_T$ is a relative Cartier divisor in spectral algebraic geometry, this is just the base change of relative Cartier divisor (Lemma 2.14).

■

We first recall a proposition in Katz and Mazur's book [KM85, Corollary 1.3.7]: Suppose that C/S is a smooth group curve, and D is a relative Cartier divisor of C, then exists a closed subscheme Z of S, satisfying for any $T \to S$, D_T is a subgroup of C_T if and only if T passing through Z.

Lemma 3.4. *Let E/R be a spectral elliptic curve, and $D \to E$ be a relative Cartier divisor. There exists a closed spectral Deligne-Mumford substack* $\mathrm{Sp\acute{e}t}Z \subset \mathrm{Sp\acute{e}t}R$*, satisfying the following universal property:*

For any $S \in \mathrm{CAlg}_R^{cn}$, such that the associated sheaf of D_S is a relative Cartier divisor of X_S and $(D_S)^{\heartsuit}$ is a subgroup of $(E_S)^{\heartsuit}$ if and only if $R \to S$ factor through Z.

Proof. For a map $R \to S$, it is obvious that D_S is a relative Cartier divisor of X_S. By [KM85, Corollary 1.3.7], we just notice that if $(D_S)^{\heartsuit}/\pi_0 S$ is a subgroup of $(E_S)^{\heartsuit}/\pi_0 S$, we have $\mathrm{Spec}\pi_0 S$ must passing through a closed subscheme $\mathrm{Spec}Z_0$ of $\mathrm{Spec}\pi_0 R$. This corresponds a closed spectral subscheme $\mathrm{Spec}Z$ of $\mathrm{Spec}R$, sice we have the map $R \to S$ such that $\pi_0 R \to \pi_0 S$ pass through $\pi_0 R/I$ for some ideal I of $\pi_0 R$, so we have $R \to S$ passing through $R^{Nil(I)}$, see [Lur18c, Chapter 7] for details about nilpotent R-module. Conversely, suppose that $R \to S$ passing through Z, then we have $S = \mathcal{O}_{\mathrm{Sp}\acute{e}tS}$ is vanishing on I. That is we have $\pi_0 R \to \pi_0 S$ passing through $\pi_0 R/\sqrt{I}$, but this is equivalent to say $\mathrm{Spec}\pi_0 S \to \mathrm{Spec}\pi_0 R$ passing through $\mathrm{Spec}\pi_0 R/I = \mathrm{Spec}Z_0$, and so $(D_S)^\heartsuit$ is a subgroup of $(E_S)^\heartsuit$.

■

Theorem 3.5. *Let E/R be a spectral elliptic curve, then the functor*

$$\begin{aligned} \mathrm{Level}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \\ & R' \mapsto \mathrm{Level}(\mathcal{A}, E_{R'}/R') \end{aligned}$$

is representable by a closed substack $S(A)$ of $\mathrm{CDiv}_{X/R}$. Moreover, $S(A) = \mathrm{Sp\acute{e}t}\mathcal{P}_{E/R}$ for an $\mathbb{E}_\infty$-ring $\mathrm{Sp\acute{e}t}\mathcal{P}_{E/R}$, which is locally almost of finite presentation over R, .

Proof. By definition, the functor $\mathrm{Level}_{E/R}$ is a subfunctor of the representable functor $\mathrm{CDiv}_{X/R}$. We consider a spectral Deligne-Mumford stack GroupCDiv defined by the pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathrm{GroupCDiv}_{E/R} & \longrightarrow & \mathrm{CDiv}_{E/R} \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}Z & \longrightarrow & \mathrm{Sp\acute{e}t}R. \end{array}$$

It is easy to say that $\mathrm{GroupCDiv}_{E/R}$ valued on a R-algebra R' is the space of relative Cartier divisors D of $E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R'$, such that $D^\heartsuit$ is a subgroup of $(E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R')^\heartsuit$. It is cleared that

$$\mathrm{GroupCDiv}_{E/R} = \coprod_{A_0 \in \mathrm{FinAb}} A_0 - \mathrm{CDiv}_{E/R}$$

where $A_0 - \mathrm{CDiv}_{E/R}$ valued on a R-algebra R' is the space of relative Cartier divisors D of $E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R'$, such that $D^\heartsuit$ is an algebric subgroup of $(E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R')^\heartsuit$ and $D^\heartsuit(R') = A_0$. It is cleared that $\mathrm{Level}_{E/R} = A - \mathrm{CDiv}_{E/R}$, so we have $\mathrm{Level}_{E/R}$ is representable by a open substack of $\mathrm{GroupCDiv}_{E/R}$.

To prove the second part, we consider the map $S(A) \to \mathrm{Sp\acute{e}t}R$, they are all spectral algebraic spaces. By [Lur18c, Remark 5.2.0.2], a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary spectral algebraic space is finite in ordinary algebraic geometry. So we only need to prove $S(A)^\heartsuit$ is finite over $\mathrm{Spec}\pi_0 R$, but this is just the classical case since $S(A)^{\heartsuit}$ is the representable object of the classical level structure, which is finite over R_0 by [KM85, Corollary 1.6.3].

■

3.2 Derived Level Structures of Spectral p-Divisible Groups

Before we talk about derived level structures of spectral p-divisible groups, let us first review something about the classical level structures of commutative finite flat group schemes. Let X/S be a finite flat S-scheme of finite presentation of rank N, it can be prove that X/S is finite locally free of rank N. This means that for every affine scheme $\mathrm{Spec}R \to S$, the pullback scheme $X \times_S \mathrm{Spec}R$ over $\mathrm{Spec}R$ have the form $\mathrm{Spec}R'$, where R' is an R-algebra which is locally free of rank N. For an element $f \in R'$ which can acts on R' by multiplication, define an R-linear endmorphism of B'. Because R' is a locally free of rank N. Then multiplication of f can be representable by a $N \times N$ matrix M_f. Then we can define the characteristic polynomial of f to be the characteristic polynomial of M_f, i.e.,

$$\det(T - f) = \det(T - M_f) = T^N - \mathrm{trace}(M_f) + \cdots + (-1)N\mathrm{Norm}(f).$$

Let $\{P_1, \cdots, P_N\}$ be a set of N points in $X(S)$, we say this set is a full set of sections of X/S if one of the following two conditions are satisfied:

1. For any $\mathrm{Spec}R \to S$, and $f \in B = H^0(X_R, \mathcal{O})$, we have the equality
$$\det(T - f) = \prod_{i=1}^{N}(T - f(p_i)).$$
2. For every $\mathrm{Spec}R \to S$, and $f \in B = H^0(X_R, \mathcal{O})$, we have
$$\mathrm{Norm}(f) = \prod_{i=1}^{N} f(p_i).$$

Actually, these conditions are equivalent.

If we have N not-necessarily-distinct points $\{P_1, \cdots, P_N\}$ in $X(S)$, then we have a morphism

$$\mathcal{O}_Z \to \bigotimes_i (P_i)_*(\mathcal{O}_S)$$

of sheave over X. It is easy to see that this map is surjective, and it defines a closed subscheme D of X, which is flat, proper over S. So by the construction, for a ϕ :

$A \to X(S)$, we can define closed subscheme D of X which corresponds to the sheave $\otimes_{a\in A}\phi(a)_*\mathcal{O}_S$.

Lemma 3.6. *For a finite flat and finite presentation S-scheme Z, $\mathrm{Hom}(A, Z)$ is an open subscheme of $\mathrm{Hilb}_{Z/S}$.*

Proof. Let $T \to S$ be a S-scheme, for any $D \to Y = T \times_S Z$ in $\mathrm{Hilb}(Y) = \mathrm{Hilb}(T \times_S Z)$, we need to prove that the set of points $t \in T$ which satisfying $D_t \to Y_t$ is coming from the closed subscheme associated with a map $\phi : A \to Z(T) = Y(T)$ is an open subset of T. Since D is the closed subscheme defined by $\mathcal{O}_Y \to \mathcal{O}_D$, if D_t comes form $\mathcal{O}_Y|_t \to \bigotimes(P_i)_*(\mathcal{O}_T)|_t$. Then by the definition of stalks of sheaves, there exists an open subset U of D such that $t \in U$, and D_U is defined by $\mathcal{O}_Y|_U \to \bigotimes(P_i)_*(\mathcal{O}_T)|_U$.

■

Definition 3.7 Suppose that G/S be a rank N commutative finite flat S-group scheme of finite presentation and A is a finite abelian group of order N. A group homomorphism

$$\phi : A \to G(S)$$

is called an A-generator of G/S, if the N points $\{\phi(a)\}_{a\in A}$ are a full subset of sections of $G(S)$. In these cases, we say ϕ is a Drinfeld level structure.

Proposition 3.8. *[KM85, Proposition 1.10.13] Suppose that G is a rank N finite flat commutative group scheme of finite presentation over S and A is a finite abelian group of order N. Then we have the following two propositions:*

1. *The functor $A-\mathrm{Gen}(G/S)$ on S-schemes defined by*

$$T \mapsto \{\phi | \phi : A \to G(T) \text{ is a Drinfeld level structure}\}$$

is representable by a finite S-scheme of finite presentation. Actually, it is the closed subscheme of $\mathrm{Hom}_{\mathrm{Sch}_S}(A, G)$ over which the image of sections $\{\phi_{univ}(a)\}_{a\in A}$ of the universal homomorphism $\phi_{univ} : A \to G$ form a full set of sections.

2. *If G/S is finite étale over S of rank N, we have*

$$A-\mathrm{Gen}(G/S) \simeq \mathrm{Isom}_{\mathrm{Sch}_S}(A, G),$$

such that each connected component of S, $A-\mathrm{Gen}(S)$ is either empty or is a finite tale $\mathrm{Aut}(A)$*-torsor.*

Derived Level Structures of Spectral Finite Flat Group Schemes

For a spectral commutative finite flat group scheme G over R. By the definition of finite flat, we have $G = \mathrm{Sp\acute{e}t}B$ for a finite flat R-algebra B. We let $\mathrm{Hilb}(G/R)$ denote the full subcategory of $\mathrm{SpDM}_{/G}$ spanned by those $D \to G$ such that $D \to G$ is a closed immersion of spectral Deligne-Mumford stacks, and the composition $D \to G \to R$ is flat, proper and locally almost of finite presentation. Then we find $\mathrm{Hilb}(G/R)$ is actually equivalent to the ∞-category of diagrams which have the form

$$\begin{array}{ccc} R & \longrightarrow & B \\ & \searrow \quad \swarrow & \\ & R' & \end{array}$$

such that R' is flat, proper and locally almost of finite presentation over R and satisfies certain conditions. It is easy to see that $\mathrm{Hilb}(G/R)$ is a Kan complex. Then we can define a functor

$$\begin{aligned} \mathrm{Hilb}_{G/R} : \mathrm{CAlg}_R^{\mathrm{cn}} &\to \mathcal{S} \\ R' &\to \mathrm{Hilb}(G_{R'}) \end{aligned}$$

Theorem 3.9. *Suppose that G is a commutative finite flat group scheme over an $\mathbb{E}_\infty$-ring R, then $\mathrm{Hilb}_{G/R}$ is representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over R.*

Proof. This is just a special case of spectral algebraic geometry version of Lurie's theorem [Lur04, Theorem 8.3.3].

■

Remark 3.10 We can proof this theorem by the same argument of the proof of representability of relative Cartier divisors.

Definition 3.11 Let G be a spectral commutative finite flat group scheme of rank N over an $\mathbb{E}_\infty$-ring R, and A be an abstract finite abelian group of order N, an A-level structure of G is an object $\phi : D \to G$ in $\mathrm{Hilb}(G/R)$, such that $\pi_0\phi_*\mathcal{O}_D \simeq \otimes\phi(a)_*\mathcal{O}_{\mathrm{Spec}\pi_0 R}$, where $\phi(a)_*\mathcal{O}_{\mathrm{Spec}\pi_0 R}$ comes from a map $\phi : A \to G^\heartsuit(\pi_0 R)$.

Lemma 3.12. *Let G/R be a spectral commutative finite flat group scheme of rank N over an $\mathbb{E}_\infty$-ring R and let D be a Hilbert closed subscheme of G. Then there exists a $\mathbb{E}_\infty$-ring Z, satisfying the following universal property:*

For any $R \to R'$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$, $(D_{R'})^\heartsuit$ is a derived A-level structures of $(G_{R'})^\heartsuit$ if and only if $R \to R'$ factor through Z.

Proof. For $R \to R'$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$, it is obvious that $D_{R'}$ is in $\mathrm{Hilb}(G_{R'}/R')$. This means that $(D_{R'})^{\heartsuit}$ is a Hilbert closed subscheme of $(G_{R'})^{\heartsuit}$. For $D_{R'}$ to be a derived level structure, we have $D_{R'}^{\heartsuit}$ must lie in $\mathrm{Hom}(A, G^{\heartsuit})(\pi_0 R')$, this means that $\mathrm{Spec}\pi_0 R' \to \mathrm{Spec}\pi_0 R$ must passing through an open of $\mathrm{Spec}\pi_0 R$, since $\mathrm{Hom}(A, G^{\heartsuit})$ can be viewed as a open sub scheme of $\mathrm{Hilb}(G^{\heartsuit}/R^{\heartsuit})$. Then we have $\pi_0 R \to \pi_0 R'$ passing through W_0, where W_0 is a localization of $\pi_0 R$, so we have $R \to R'$ must passing through W, where W is an $\mathbb{E}_\infty$-ring, which is a localization of R. As for now, we already have a map $\mathrm{Sp\acute{e}t}R' \to \mathrm{Sp\acute{e}t}W$, such that $D_{R'}$ is a Hilbert closed subscheme of $G_{R'}$, and $\pi_0 i_* \mathcal{O}_{D_{R'}}$ comes from a map $\phi : A \to G^{\heartsuit}(\pi_0 R')$. For $D_{R'}$ want to be a derived level structure, $\mathcal{O}_{G^{\heartsuit}} \to \phi(a)_*(\mathcal{O}_{\mathrm{Spec}\pi_0 R'})$ needs to be an isomorphism, i.e., these N points $\phi(a)_{a\in A}$ must be a full section of $G^{\heartsuit}(\pi_0 R')$. By [KM85, Proposition 1.9.1], for a set of N points of $(G^{\heartsuit}(\pi_0 R'))$ to be a full section of $G^{\heartsuit}(\pi_0 R')$, $\mathrm{Spec}\pi_0 R' \to \mathrm{Spec}\pi_0 W$ must passing through a closed subscheme of $\mathrm{Spec}W_0$. Then $\pi_0 W \to \pi_0 R'$ must passing through Z_0, where Z_0 is equals $\pi_0 W/I$ for some ideal I of $\pi_0 W$. This means that we have $W \to R'$ pass through $Z = W^{\mathrm{Nil}(I)}$. By the discussion above, we have Z is the desired $\mathbb{E}_\infty$-ring. And the converse is also true by the same discussion in the derived level structures of curves.

■

Proposition 3.13. *Suppose that G is a spectral commutative finite flat group scheme of rank N over an $\mathbb{E}_\infty$-ring R and A is an abstract finite abelian group of order N. Then the following functor*

$$\mathrm{Level}_{H/R}^{\mathcal{A}} : \mathrm{CAlg}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}(\mathcal{A}, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S(A) = \mathrm{Sp\acute{e}t}\mathcal{P}_{G/R}$.

Proof. We first prove the representability. By definition, the functor $\mathrm{Level}_{G/R}^{\mathcal{A}}$ is a subfunctor of the representable functor $\mathrm{Hilb}_{G/R}$. We consider a spectral Deligne-Mumford stack $S(A)$ defined by the pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} S(A) & \longrightarrow & \mathrm{Hilb}_{G/R} \\ \downarrow & & \downarrow \\ \mathrm{Sp\acute{e}t}Z & \longrightarrow & \mathrm{Sp\acute{e}t}R. \end{array}$$

It is easy to say that $S(A)$ valued on a R-algebra R' is the Hilbert closed subscheme D of $E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R'$, such that $D^{\heartsuit}$ is a derived level A-structure of $(E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}R')^{\heartsuit}$. Then $S(A)$ is the desried stack.

For the affine condition, we need to prove that $S(A)$ is finite in spectral algebraic geometry. By [Lur18c, Remark 5.2.0.2], a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary spectral algebraic space is finite in ordinary algebraic geometry. We have $S(A)$ and $\mathrm{Sp\acute{e}t}R$ are spectral spaces. So we only need to prove $S(A)^{\heartsuit}$ is finite over R_0, but this is just the classical case, which is finite by [KM85, Proposition 1.10.13].

■

Derived Level Structures of Spectral p-Divisible Groups

Remark 3.14 We let $\mathrm{FFG}(R)$ denote the ∞-category of spectral commutative finite flat group schemes over an $\mathbb{E}_\infty$-ring R. By [Lur18a, Proposition 6.5.8], there is another equivalent definition of spectral p-divisible group [Lur18b, Definition 6.0.2]. A spectral p-divisible group over a connective $\mathbb{E}_\infty$-ring R is just a functor

$$G : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$$

which satisfies the following conditions:

1. Suppose that $S \in \mathrm{CAlg}_R^{cn}$, the spectrum $G(S)$ is p-nilpotent, i.e., $G(S)[1/p] \simeq 0$.
2. For M be a finite ableian p-group, the functor

$$\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad S \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbb{Z}}}(M, G(S))$$

   is corepresentable by a finite flat R-algebra.

Let X be a spectral p-divisible group of height h over an $\mathbb{E}_\infty$-ring R, that is a functor

$$X : \mathrm{Ab}_{\mathrm{fin}}^{p} \to \mathrm{FFG(R)}.$$

For every $p^k \in \mathrm{Ab}_{\mathrm{fin}}^{p}$, we let $X[p^k]$ denote the image of p^k of X. We find that $X[p^k]$ is a rank $(p^k)^h$ spectral commutative finite flat group schemes over R.

Definition 3.15 Let G be a spectral p-divisible group of height h over a connective E_∞-ring R . For A a finite abelian group, an derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure

$$\phi : D \to G[p^k]$$

of $G[p^k]$, which is a spectral commutative finite flat scheme over R. We let $\mathrm{Level}(k, G/R)$ denote the ∞-groupoid of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structures of G/R.

Theorem 3.16. *Let G be a spectral p-divisible group of height h over an $\mathbb{E}_\infty$-ring R. Then the following functor*

$$\mathrm{Level}_{G/R}^{k} : \mathrm{CAlg}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}(k, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S(k) = \mathrm{Sp\acute{e}t}\mathcal{P}^k_{G/R}$.

Proof. We just notice that by the definition of spectral p-divisible group, $G[p^k]$ is a spectral commutative finite flat scheme. Then the theorem follows form the above result of general spectral commutative finite flat group scheme.

■

Non-Full Level Structures

The above cases only cares full level structures of commutative finite flat schemes, actually we can define general level structures of finite flat group schemes. Let G be a spectral commutative finite flat group scheme of rank N over an $\mathbb{E}_\infty$-ring R, and A be an abstract finite abelian group, an derived A-level structure of G is an object $\phi : D \to G$ in $\mathrm{Hilb}(G/R)$, such that $D^\heartsuit$ is a subgroup of G and $G^\heartsuit(\pi_0 R)$ is isomorphic to A. We let $\mathrm{Level}_1(\mathcal{A}, G/R)$ denote th space of derived A-level structure. And $\mathrm{Level}_0(\mathcal{A}, G/R)$ denote the space of equivalence class $D \to G$ in $\mathrm{Hilb}(G/R)$ such that $G^\heartsuit(\pi_0 R)$ is isomorphic to A, two object D, D' are equivalent if the image of $D^\heartsuit \to G^\heartsuit$ and $D'^\heartsuit \to G^\heartsuit$ are same.

Proposition 3.17. *Suppose that G is a spectral commutative finite flat group scheme of rank N over an $\mathbb{E}_\infty$-ring R and A is an abstract finite abelian group of order not necessarily equal to N. Then the following functor*

$$\mathrm{Level}^{1,\mathcal{A}}_{G/R} : \mathrm{CAlg}^{cn}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}_1(\mathcal{A}, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack.

Proof. We just notice that the classical level structure functor $\mathrm{Level}(A, G^\heartsuit/\pi_0 R)$ is representable by a closed subscheme $\mathrm{Hom}(A, G)$, the using the same discussion of full level case, we get the desired result.

■

Remark 3.18 The above proposition also true for $\mathrm{Level}^{0,\mathcal{A}}$. By the spectral commutative finite flat scheme cases, we can get the representability results of spectral p-divisible group case.

We let $\mathrm{Level}_1(k, G/R)$ denote the ∞-groupoid of derived $(\mathbb{Z}/p^k\mathbb{Z})$-level structures of G/R. Then the following functor

$$\mathrm{Level}^{1,k}_{G/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}_1(k, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S_1(k) = \mathrm{Sp\acute{e}t}\mathcal{P}^{1,k}_{G/R}$.

We let $\mathrm{Level}_0(k, G/R)$ denote the ∞-groupoid of derived $(\mathbb{Z}/p^k\mathbb{Z})$-level generators of G/R. Then the following functor

$$\mathrm{Level}^{0,k}_{G/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}_0(k, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S_0(k) = \mathrm{Sp\acute{e}t}\mathcal{P}^{0,k}_{G/R}$.

4 Applications

4.1 Spectral Elliptic Curves with Derived Level Structures

There exists a spectral Deligne-Mumford stack $\mathcal{M}_{ell}$ whose functor of points is

$$\begin{aligned} \mathcal{M}_{ell} \ : \ & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{ell}(R), \end{aligned}$$

where $\mathcal{M}_{ell}(R) = \mathrm{Ell}(R)^{\simeq}$ is the underline ∞-groupoid of the ∞-category of spectral elliptic curves over R.

And we have the classical Deligne-Mumford stack of classical elliptic curves, which can be viewed as a spectral Deligne-Mumford stack

$$\begin{aligned} \mathcal{M}^{cl}_{ell} \ : \ & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}^{cl}_{ell}(\pi_0 R) \end{aligned}$$

where $\mathcal{M}^{cl}_{ell}(\pi_0 R)$ is the groupoid of classical elliptic curves over the commutative ring $\pi_0 R$.

And for A denote $\mathbb{Z}/N\mathbb{Z}$, or $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, we have the classical Deligne-Mumford stack of classical elliptic curves with level-A structures, which can also be viewed as a spectral Deligne-Mumford stack.

$$\begin{aligned} \mathcal{M}^{cl}_{ell}(A) \ : \ & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}^{cl}_{ell}(A)(\pi_0 R) \end{aligned}$$

where $\mathcal{M}^{cl}_{ell}(A)(\pi_0 R)$ is the groupoid of classical elliptic curves with level A-structures over the commutative ring $\pi_0 R$.

In last chapter, we define and study derived level structures. The construction $X \mapsto \mathrm{Level}(\mathcal{A}, X/R)$ determines a functor $\mathrm{Ell}(R) \to \mathcal{S}$ which is classified by a left fibration $\mathrm{Ell}(\mathcal{A})(R) \to \mathrm{Ell}(R)$. Objects of $\mathrm{Ell}(\mathcal{A})(R)$ are pairs (E, ϕ), where E is a spectral elliptic curve and ϕ is a derived level structures of E.

For every $R \in \mathrm{CAlg}^{\mathrm{cn}}$, we can consider all spectral elliptic curves over R with derived level structures. This moduli problem can be thought as a functor

$$\begin{aligned}\mathcal{M}_{ell}(\mathcal{A}) \quad : \quad & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{ell}(\mathcal{A})(R) = \mathrm{Ell}(\mathcal{A})(R)\end{aligned}$$

where $\mathrm{Ell}(\mathcal{A})(R)$ is the space of spectral elliptic curves E with a derived level structure $\phi : \mathcal{A} \to E$.

Proposition 4.1. *The functor $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{\mathrm{cn}} \mapsto \mathcal{S}$ is an étale sheaf.*

Proof. Let $\{R \to U_i\}$ be an étale cover of R, and $U_\bullet$ be the associate check simplicial object. We consider the following diagram

$$\begin{array}{ccc}\mathrm{Ell}(\mathcal{A})(R)^{\simeq} & \xrightarrow{f} & \lim_{\Delta} \mathrm{Ell}(\mathcal{A})(U_\bullet)^{\simeq} \\ \downarrow p & & \downarrow q \\ \mathrm{Ell}(R)^{\simeq} & \xrightarrow{g} & \lim_{\Delta} \mathrm{Ell}(U_\bullet)^{\simeq}.\end{array}$$

The left map p is a left fibration between Kan complex, so is a Kan fibration [Lur09a, Lemma 2.1.3.3]. And the right vertical map is pointwise Kan fibration. By picking a suit model for the homotopy limit we may assume that q is a Kan fibration as well. We have g is an equivalence by [Lur18a, Lemma 2.4.1]. To prove that f is a equivalence. We only need to prove that for every $E \in \mathrm{Ell}(R)$, the map

$$p^{-1}E \simeq \mathrm{Level}(\mathcal{A}, E/R) \to \lim_{\Delta} \mathrm{Level}(\mathcal{A}, E \times_R U_\bullet/U_\bullet) \simeq q^{-1}g(E)$$

is an equivalence. We have the $\mathrm{Level}(\mathcal{A}, E)$ as full ∞-subcategory of $\mathrm{CDiv}(E/R)$ and $\lim_{\Delta} \mathrm{Level}(\mathcal{A}, E \times_R U_\bullet)$ as a full subcategory of

$$\lim_{\Delta} \mathrm{CDiv}(E \times_R U_\bullet(U_\bullet))$$

But CDiv is an tale sheaf. So the functor

$$\mathrm{Level}(\mathcal{A}, E/R) \to \lim_{\Delta} \mathrm{Level}(\mathcal{A}, E \times_R U_\bullet/U_\bullet).$$

is fully faithful. To prove it is a equivalence, we only need to prove it is essentially surjective.

For any $\{\phi_{U_\bullet} : D \to E \times_R U_\bullet\}$ in $\lim_{\Delta} \mathrm{Level}(\mathcal{A}, E \times_R U_\bullet/U_\bullet)$. Clearly, we can find a morphism $\phi_R : D \to E$ in $\mathrm{CDiv}(E/R)$ whose image under the equivalence $\mathrm{CDiv}(E/R) \simeq \lim_{\Delta} \mathrm{CDiv}(E \times_R U_\bullet/U_\bullet)$ is $\{\phi_{U_\bullet} : D \to E \times_R U_\bullet\}$. We just need to prove this $\phi_R : D \to E$

is a derived level structure. This is true since in the classic case, $\mathrm{Level}(A, E^{\heartsuit}(R_0)) \simeq \lim_{\Delta} \mathrm{Level}(A, E^{\heartsuit}(\tau_{\leq 0}U_{\bullet}))$ and $\phi_R : D \to E$ is already a relative Cartier divisor.

■

Lemma 4.2. *$\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is a nilcomplete functor, i.e., $\mathcal{M}_{ell}(\mathcal{A})(R)$ is the homotopy limit of the following diagram*

$$\cdots \to \mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq m}R) \to \mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq m-1}R) \to \cdots \to \mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq 0}R)$$

Proof. For a spectral elliptic curve R, there is an obvious functor

$$\theta : \mathcal{M}_{ell}(\mathcal{A})(R) \to \varprojlim_{n} \mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq n}R)$$

define by $(E, \phi : D \to E) \mapsto \{(E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R, \phi_n : D \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R \to E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R)\}_n$. Here we notice that $(E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R, \phi_n : D \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R \to E \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R)$ is in $\mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq n}R)$.

First, we prove that θ is essentially surjective. An object in $\varprojlim_{m} \mathcal{M}_{ell}(\mathcal{A})(\tau_{\leq m}R)$ can be written as a diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & D_{n+1} & \longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots & \longrightarrow & D_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & E_{n+1} & \longrightarrow & E_n & \longrightarrow & E_{n-1} & \longrightarrow & \cdots & \longrightarrow & E_0 \end{array}$$

where each E_n is spectral elliptic curve over $\tau_{\leq n}R$ and $D_n \to E_n$ is a derived level structure, and satisfying $D_n = D_{n+1} \times_{\mathrm{Sp\acute{e}t}\tau_{\leq n+1}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R, E_n = E_{n+1} \times_{\mathrm{Sp\acute{e}t}\tau_{\leq n+1}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R$. By the nilcompletness of $\mathcal{M}_{ell}$, we get a spectral elliptic curves E, such that $E \times_R \tau_{\leq n}R \simeq E_n$, and by the nilcompletness of Var_+ [Lur18c, Proposition 19.4.2.1], we get a spectral Deligne-Mumford stack D, such that $D_n = D \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}\tau_{\leq n}R$. We need to prove the induce map $D \to E$ is a derived level structure, but this follows form nilcompletness of $\mathrm{Level}_{E/R}$.

Second, we need to prove that this functor is fully faithful. Unwinding the definitions, we need to prove that for every $(X, D_1 \to X), (Y, D_2 \to Y) \in \mathcal{M}_{ell}(\mathcal{A})(R)$, the following map is a homotopy equivalence.

$$\mathrm{Map}_{\mathcal{M}_{ell}(\mathcal{A})(R)}((X, D_X), (Y, D_Y)) \to \mathrm{Map}_{\mathcal{M}_{ell}(\mathcal{A})(R)}(\varprojlim_{n}(X_n, D_{X,n}), \varprojlim_{n}(Y_m, D_{Y,m})).$$

where X_n is $\tau_{\leq n}X = X \times_R \tau_{\leq n}R$, and Y, $D_{X,n}, D_{Y,n}$ similarly.

But we notice that this is equivalent to following equivalence

$$\mathrm{Map}_{\mathrm{SpDM}_{/R}}((X, D_X), (Y, D_Y)) \to \varprojlim_{n} \mathrm{Map}_{\mathrm{SpDM}_{\tau_{\leq n}}}((X_n, D_{X,n}), (Y_n, D_{Y,n})).$$

And this equivalence follows from [Lur18c, Proposition 19.4.1.2]

■

Lemma 4.3. *$\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is a cohesive functor.*

Proof. For every pullback diagram

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ such that the underlying homomorphisms $\pi_0 A \to \pi_0 B \leftarrow \pi_0 C$ are surjective. We need to prove that

$$\begin{array}{ccc} \mathcal{M}_{ell}(\mathcal{A})(D) & \longrightarrow & \mathcal{M}_{ell}(\mathcal{A})(A) \\ \downarrow & & \downarrow \\ \mathcal{M}_{ell}(\mathcal{A})(C) & \longrightarrow & \mathcal{M}_{ell}(\mathcal{A})(B) \end{array}$$

is a pullback diagram.

We have the following diagram in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$,

$$\begin{array}{ccc} \mathcal{M}_{ell}(\mathcal{A}) & \xrightarrow{g} & \mathcal{M}_{ell} \\ & \searrow^{f} & \downarrow h \\ & & * \end{array}$$

By [Lur18c, Remark 17.3.7.3], $\mathcal{M}_{ell} * (\mathcal{A})$ is a cohesive fucntor if and only if f is cohesive. Since we have $\mathcal{M}_{ell}$ is cohesive functor, h is a cohesive morphism in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$. And again by [Lur18c, Remark 17.3.7.3], f is cohesive if and only if g is cohesive. So we only need to prove that g is a cohesive morphism. But by [Lur18c, Proposition 17.3.8.4] g is cohesive if and only if each fiber of g is cohesive, i.e., for $R \in \mathrm{CAlg}^{cn}$ and a point $\eta_E \in \mathcal{M}_{ell}(R)$ which represents a spectral elliptic curve E, the functor

$$f_E : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\}$$

is cohesive. But we have $R' \mapsto \mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\} \simeq \mathrm{Level}(\mathcal{A}, E \times_R R'/R') \simeq \mathrm{Level}_{E/R}(R')$. The cohesive of $\mathcal{M}_{ell}(\mathcal{A})$ then follows from the cohesive of $\mathrm{Level}_{E/R}$.

■

Lemma 4.4. *The fucntor $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is integrable*

Proof. We need to prove that for R a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal $m \subset \pi_0 R$, then there is an equivalence

$$\mathrm{Map}_{Fun(\mathrm{CAlg}^{cn}, \mathcal{S})}(\mathrm{Sp\acute{e}t} R', \mathcal{M}_{ell}(\mathcal{A})) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})}(\mathrm{Spf} R', \mathcal{M}_{ell}(\mathcal{A})).$$

We have the following diagram in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$,

$$\begin{array}{ccc} \mathcal{M}_{ell}(\mathcal{A}) & \xrightarrow{g} & \mathcal{M}_{ell} \\ & \searrow^{f} & \downarrow h \\ & & * \end{array}$$

By [Lur18c, Remark 17.3.7.3], $\mathcal{M}_{ell}(\mathcal{A}) \to *$ is a integrable fucntor if and only if f is integrable. Since we have $\mathcal{M}_{ell}$ is integrable functor, h is a integrable morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. And again by [Lur18c, Remark 17.3.7.3], f is integrable if and only if g is integrable. So we only need to prove that g is a integrable morphism. But by [Lur18c, Proposition 17.3.8.4] g is integrable if and only if each fiber of g is integrable, i.e., for $R \in \mathrm{CAlg}^{\mathrm{cn}}$ and a point $\eta_E \in \mathcal{M}_{ell}(R)$ which represents a spectral elliptic curve E, the functor

$$f_E : \mathrm{CAlg}_R^{cn} \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\}$$

is integrable. But we have $R' \mapsto \mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\} \simeq \mathrm{Level}(\mathcal{A}, E \times_R R'/R') \simeq \mathrm{Level}_{E/R}(R')$. The integrable of $\mathcal{M}_{ell}(\mathcal{A})$ then follows from the integrable of $\mathrm{Level}_{E/R}$.

■

Lemma 4.5. *The functor $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{\mathrm{cn}} \mapsto \mathcal{S}$ admits a cotangent complex $L_{\mathcal{M}_{ell}^{\mathrm{de}}}$, and moreover $L_{\mathcal{M}_{ell}^{\mathrm{de}}}$ is connective and almost perfect.*

Proof. We have a commutative diagram in $\mathrm{CAlg}^{cn} \to \mathcal{S}$,

$$\begin{array}{ccc} \mathcal{M}_{ell}(\mathcal{A}) & \xrightarrow{g} & \mathcal{M}_{ell} \\ & \searrow^{f} & \downarrow h \\ & & * \end{array}$$

Since we have h is infitessimally coheisve and admits a connective cotangent complex, and f,g is infitessimally cohesive. By [Lur18c, Proposition 17.3.9.1], to prove that f admits a cotangent complex. We only need to prove g admits a relative cotangent complex. By [Lur18c, Proposition 17.2.5.7], a morphism $j : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$ admits a relative contangent complex if and only if, for any corepresentbale $Y' = \mathrm{Map}(R, -) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ and any natural transformation $Y' \to U$, j' in the following pullback diagram admit a cotangent complex.

$$\begin{array}{ccc} Y' \times_Y X & \longrightarrow & X \\ \downarrow j' & & \downarrow j \\ Y' & \longrightarrow & Y \end{array}$$

To prove that $\mathcal{M}_{ell}(\mathcal{A}) \to \mathcal{M}_{ell}$ admits a cotangent a cotangent complex, we just need to prove that for any $R \in \mathrm{CAlg}^{cn}$, and a spectral elliptic curve E which represents a natural transformations $\mathrm{Spec} R \to \mathcal{M}_{ell}$. The functor

$$\mathrm{CAlg}_R \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\}$$

admits a connective cotangent complex. But we have $\mathcal{M}_{ell}(\mathcal{A})(R') \times_{\mathcal{M}_{ell}(R')} \{\eta_E\} = \mathrm{Level}(E \times_R R') = \mathrm{Level}_{E/R}(R')$. So the results of $f : \mathcal{M}_{ell}(\mathcal{A}) \to *$ admits a cotangent complex follows from $\mathrm{Level}_{E/R}$ admits a cotangent complex. And the properties of connective and almost perfect also follows from the property of the cotangent complex of $\mathrm{Level}_{E/R}$.

■

Lemma 4.6. *The functor $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \mapsto \mathcal{S}$ is locally almost of finite presentation.*

Proof. Consider the functor $\mathcal{M}_{ell}(\mathcal{A}) \to *$, it is infitesimally cohesive and admits a cotangent complex which is almost perfect, so by [Lur18c, 17.4.2.2], it is locally almost of finite presentation. So $\mathcal{M}_{ell}(\mathcal{A})$ is locally almost of finite presentation, since $*$ is a final object of $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$.

■

Theorem 4.7. *The functor*

$$\begin{aligned} \mathcal{M}_{ell}(A) \quad : \quad & \mathrm{CAlg} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{ell}(\mathcal{A})(R) = \mathrm{Ell}(\mathcal{A})(R)^{\simeq} \end{aligned}$$

is representable by a spectral Deligne-Mumford stack.

Proof. By the spectral Artin representability theorem, we need to prove that the functor $\mathcal{M}_{ell}(\mathcal{A})$ satisfying the following condition

1. The space $\mathcal{M}_{ell}(\mathcal{A})(R_0)$ is n-truncated for every discrete commutative ring R_0.
2. $\mathcal{M}_{ell}(\mathcal{A})$ is a sheaf for the étale topology.
3. $\mathcal{M}_{ell}(\mathcal{A})$ is a nilcomplete, infinitesimally cohesive, and integrable functor.
4. $\mathcal{M}_{ell}(\mathcal{A})$ admits a cotangent complex $L_{\mathcal{M}_{ell}(A)}$ which is connective.
5. $\mathcal{M}_{ell}(\mathcal{A})$ is locally almost of finite presentation.

But these follows form the above series of lemmas.

■

4.2 Higher Categorical Lubin-Tate Towers

We recall that for a height h p-divisible group G_0 over a commutative ring R_0 and suppose $A \in \mathrm{CAlg}^{ad}_{cpl}$. We recall that a deformation of G_0 over R is a spectral p-divisible group over R together with an equivalence class of G_0-tagging of G. We let $\mathrm{Level}(k, G/R)$ denote the space of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of a height h spectral p-divisible group. We consider the following functor

$$\begin{aligned} \mathcal{M}_k \ : \ & \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}(G_0, R, k)$ is the ∞-category whose objects are triples (G, ρ, η)

1. G is a spectral p-divisible group over R.

2. ρ is an equivalence of G_0 taggings of R.

3. $\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G.

Theorem 4.8. *The functor $\mathcal{M}_k$ is corepresentable by a $\mathbb{E}_\infty$-ring which is finite over the unoriented spectral deformation ring of G_0.*

Proof. We let $E_{univ}/R^{\mathrm{un}}_{G_0}$ denote the universal spectral deformation of G_0/R_0. Suppose that G is a spectral deformation G_0 to R, we get a map of $\mathbb{E}_\infty$-rings $R^{\mathrm{un}}_{G_0} \to R$, and an equivalence $E_{univ} \times_{R^{un}_{G_0}} R \simeq G$ of spectral p-divisible groups. By the universal objects of level structures. We have the following equivalence

$$\mathrm{Level}(k, G/R) \simeq \mathrm{Level}(k, E_{univ} \times_{R^{un}_{G_0}} R) \simeq \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R^{un}_{G_0}}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R),$$

where $\mathcal{P}_{E_{univ}/R^{un}_{G_0}}$ is the universal object of derived level structure functor associated with the p-divisible group $E_{univ}/R^{un}_{G_0}$.

Then we consider the following moduli problem

$$\mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S}, \quad R \mapsto \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R).$$

For $R \in \mathrm{CAlg}^{ad,cpl}_{R_0}$, $\mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R)$ can viewed the ∞-categories of pairs (α, f), where

$$\alpha : R^{un}_{G_0} \to R$$

is the classified map of a spectral p-divisible group G, which is a deformation of G_0, that is $\alpha = (G, \rho)$, and $f \in \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R^{un}_{G_0}}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R) = \mathrm{Level}(k, E_{univ} \times_{R^{un}_{G_0}} R)$ is a derived level structure of G/R. So we get $\mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R)$ is just the ∞-category of pairs (G, ρ, η). By lemma 3.16, $\mathcal{P}_{E_{univ}/R^{un}_{G_0}}$ is finite over $R^{un}_{G_0}$. So we have $\mathcal{P}_{E_{univ}/R^{un}_{G_0}}$ is the desired spectrum.

■

Although we get spectra come from a conceptual derived moduli problems, but these spectra may be complicated, since we didn't know the homotopy groups. In algebraic topology, orientation of $\mathbb{E}_\infty$-spectra make E_2 page of Atiyah-Hirzebruch spectral sequences degenerating, and give us the information of homotopy groups.

Let G_0 be a height h p-divisible group over R_{G_0}. We consider the following functor

$$\begin{aligned} \mathcal{M}_k^{or} \ &: \ \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ &\quad R \to \mathrm{DefLevel}^{or}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, k)$ is the space of four tuples (G, ρ, e, η), where

1. G is a spectral p-divisible over R.

2. ρ is an equivalence class of G_0 taggings of R.

3. $e : S^2 \to \Omega^\infty G^\circ(R)$ is an orientation of the G°, where G° is the identity component of G.

4. $\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G.

Theorem 4.9. *The functor $\mathcal{M}_k^{or} : \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S}$ is corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JK}_k$, which is finite over the orientated deformations ring $R^{or}_{G_0}$.*

Proof. Let $\mathrm{Def}^{or}(G_0, R)$ denote the ∞-groupoid of triples (G, ρ, e), where G is a p-divisible of over R, ρ is an equivalence class of G_0-taggings of R, and e is an orientation of the identity conpoment of G. By [Lur18b, Theorem 6.0.3, Remark 6.0.7], the functor

$$\begin{aligned} \mathcal{M}^{or} \ &: \ \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ &\quad R \to \mathrm{Def}^{or}(G_0, R) \end{aligned}$$

is corepresnetable by the orientated deformation ring $R^{or}_{G_0}$, that is we have an equivalence of spaces

$$\mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R^{or}_{G_0}, R) \simeq \mathrm{Def}^{or}(G_0, R).$$

Let E^{or}_{univ} be the associated universal orientation deformation of G_0 to $R^{or}_{G_0}$, then it is obvious that $\mathcal{JL}_k = \mathcal{P}_{E^{or}_{univ}/R^{or}_{G_0}}$, the universal object of derived level structures of $E^{or}_{univ}/R^{or}_{G_0}$, is the desired spectrum similar to th unorientated case.

■

We call this spectrum $\mathcal{JL}_k$ the Jacquet-Langlands spectrum. It is easy to see that this $\mathcal{JL}_k$ admit an action of $GL_h(\mathbb{Z}/p^k\mathbb{Z}) \times \mathrm{Aut}(G_0)$. And when k varies, we have a tower

$$\begin{array}{c} \mathrm{Sp\acute{e}t}\mathcal{JL}_k \\ \downarrow \\ \mathrm{Sp\acute{e}t}\mathcal{JL}_{k-1} \\ \downarrow \\ \cdots \\ \downarrow \\ \mathcal{JL}_0. \end{array}$$

We call this tower higher categorical Lubin-Tate tower.

Let E be a local field, G be a reductive group over E. The classical local Langlands correspondence predict that for any irreducible smooth representation π of $G(E)$, we can naturally associate an L-parameter

$$\phi_E : W_E \to \widehat{G}(\mathbb{C}).$$

The geometric Langlands correspondence actually aim to construct an equivalence of categories

$$D(\mathrm{QCoh}(\mathrm{LocSys}_{G^\vee}(X)) \simeq D(\mathcal{D}(\mathrm{Bun}_G))$$

from the derived category of quasi-coherent sheaves on $G^\vee$ local systems on X and the derived categories of D-modules on the moduli stack of G-bundles over X [BD91]. Due to the work of Fargues-Scholze [FS21], the arithmetic local Langlands correspondence can also be some kinds of geometric Langlands correspondence, but in the perfectoid world.

In the classical arithmetic geometry, the Lubin-Tate tower can be used to realize the Jacquet-Langlands correspondence [HT01]. Is there a topological realization of the Jacquet-Langlands correspondence? Actually, in a recent paper [SS23], they already realized a version of topological Jacquet-Langlands correspondence. But their method is based on the Goerss-Hopkins-Miller-Lurie sheaf. They actually consider the degenerate level structures such that representing object is étale over representing object of universal deformations.

We hope our higher categorical analogues of Lubin-Tate towers can also establish a topological version of the classical Langlands correspondence, which means that we construct representations on the category of spectra. By the construction of Jacquet-Langlands spectra above, Let $\mathbb{G}$ be a formal group over a field of characteristic p, $\mathcal{JL}$ be its ℓ-adic complete Jacquet-Langlands spectrum. Let X be a spectrum with an action of $\mathrm{Aut}(\mathbb{G}_h)$. We have the following brave conjecture.

Conjecture 4.10. *The function spectrum $F(X, \mathcal{JL})$ admits an action of $GL_h(\mathbb{Z}_p)$ and all its homotopy groups are $\mathbb{Z}_l$-modules.*

4.3 Topological Lifts of Power Operation Rings

We recall the deformation of formal groups. Let G_0 be a formal group over a perfect field k such that $\text{char} k = p$, a deformation of G_0 to R is a triple (G, i, Φ) satisfying

- G is a formal group over R,
- There is a map $i : k \to R/m$
- There is an isomorphism $\Phi : \pi^* G \cong i^* G_0$ of formal groups over R/m.

Suppose that we have a complete local ring R whose residue filed has characteristic p. Let $\phi : R \to R, x \mapsto x^p$ be the Frobenius map. For each formal group G over R, the **Frobenius isogeny** $\text{Frob} : G \to \phi^* G$ is the homomorphism of formal group over R induced by the relative Frobenius map on rings. We write $\text{Frob}^r : G \to (\phi^r)^* G$ which is the composition $\phi^*(\text{Frob}^{r-1}) \circ \text{Frob}$

Let G_0 be a formal group over k, (G, i, α) and (G', i', α') be two deformations of G_0 to R. A deformation of Frob^r is a homomorphism $f : G \to G'$ of formal groups over R which satisfying

1. $i \circ \phi^r = i'$ and $i^*(\phi^r)^* G_0 = (i')^* G_0$.

$$\begin{array}{ccc} k & \xrightarrow{i'} & R/m \\ {\scriptstyle \phi^r}\downarrow & \nearrow{\scriptstyle i} & \\ k & & \end{array}$$

2. the square

$$\begin{array}{ccc} i^* G_0 & \xrightarrow{i^*(\text{Frob}^r)} & i^*(\phi^r)^* G_0 \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ \pi^* G & \xrightarrow{\pi^*(f)} & \pi^* G' \end{array}$$

of homomorphisms of formal groups over R/m commutes.

We let Def_R denote the category whose objects are deformations fo G_0 to R, and whose morphisms are deformation of Frob^r for some $r \geq 0$. We will say that a morphism in Def_R has height r, if it is a deformation of Frob^r, and the we denote the corresponding subcategory as $\text{Sub}^r R$. Let G be deformation of G_0 to R, then it can be proved that the assignment $f \to \text{Ker} f$ is a one-to-one correspondence between the morphisms in Sub^r_R with source G and the finite subgroup of G which have rank p^r.

Theorem 4.11. *[Str97] Let G_0/k be a height n formal group over a perfect field k. For each $r > 0$, there exists a complete local ring A_r which carries a universal height r morphism $f^r_{univ} : (G_s, i_s, \alpha_s) \mapsto (G_t, i_t, \alpha_t) \in \mathrm{Sub}^r(A_r)$. That is the operation $f^r_{univ} \to g^*(f^r_{univ})$ define a bijective relation from the set of local homomorphism $g : A_r \to R$ to the set Sub^r_R. Furthermore, we have:*

1. *$A_0 \approx W(k)[[v_1, \cdots, v_{n-1}]]$ is the Lubin-Tate ring.*
2. *There is a map $s : A_0 \to A_r$ which classifies the source of the universal height r map, i.e. $G_s = s^* G_E$, where $G_E = G_{univ}/A_0$ be the universal deformation of G_0, and A_r is finite and free as an A_0 module.*
3. *There is a map $t : A_0 \to A_r$ which classifies the target of the universal height r map, i.e. $G_t = t^* G_E$.*
4. *And there is a bijection $\{g : A_r \to R\} \to \mathrm{Sub}^r(R)$ given by $g \to g^*(f^r_{univ})(g^* G_s \to g^* G_t)$.*

We know that those rings $A_r, r \geq 0$ have topological meansings.

Theorem 4.12. *[Str98] The ring A_r in the universal deformation of Frobenuis is isomorphic to $E^0(B\Sigma_{p^r})/I$, i.e,*

$$A_r \cong E^0(B\Sigma_{p^r})/I$$

where I is transfer ideal.

The collections $\{A_r\}$ have the structures of graded coalgerbas, for $s = s_k, t = t_k : A_0 \to A_k$, which is induced by E^0 cohomology on $B\Sigma \to *$, we have

$$\mu = \ mu_{k,l} : A_{k+l} : A_{k+l} \to A_k{}^s \otimes_{A_0}{}^t A_l$$

which classifying the source,target, and composite of morphisms. So for the power operation $R^k(X) \to R^k(X \times B\Sigma_m)$. For $x = *$, we have

$$\pi_0 R \to E^0(B\Sigma_{p^r})/I \otimes \pi_0 R = A[r] \otimes \pi_0 R$$

This make $\pi_0 R$ becomes a Γ-module, where Γ are duals of $A[r]$.

For more details about power operation in Morava E-theory, one can see [Rez06, Rez09] and [Rez13]. Direct computations are in [Rez08] for height 2 at the prime 2, [Zhu14] for height 2 at prime 3, [Zhu19] for height 2 at all primes. Cases of height > 2 is still lack of computations.

Because we have the assignment $f \to \mathrm{Ker} f$ is a one-to-one correspondence between the morphisms in Sub^r_R with source G and the finite subgroup of G which have rank p^r. So it is easy to see that A_r corepresent the following moduli problem

$$\mathcal{M}_{0,r} \quad : \quad \mathrm{CAlg}_k^{\heartsuit} \to \mathcal{S}$$
$$R \to \mathrm{Def}(G_0, R, p^r)$$

where $\mathrm{Def}(G_0, R, p^r)$ consists of pairs (G, H) where G is an defomration G_0 to R, and H is a rank p^r subgroup of G.

Proposition 4.13. *For every integer $r \geq 1$, there exists a E_∞-ring $E_{n,r}$, such that $\pi_0 E_{n,r} = A_r$.*

Proof. For the formal group G_0 over a field k of characteristic p. We just consider the functor $\mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S}$ by sending an E_∞-ring R to quadruples (G, ρ, e, η), where (G, ρ) is spectral deformation of G_0 to R. e is an orientation of G°, the identity component G, and $\eta \in \mathrm{Level}_0(k, G/R)$ is a derived level structure. Using the same argument in full level structure and the fact $\mathrm{Level}_{G/R}^{0,k}$ is representable, see Remark 3.18. We get this proposition.

■

Remark 4.14 Although, we obtain spectra whose π_0 are the power operation rings of Morava E-theories. But we don't know higher homotopy groups of these spectra, since these spectra are not even periodic and they are not étale over Morava E-theories. We will continue to study such spectra in the future.

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