Derived Moduli Problems and New Cohomology Theories

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- Spectral Algebraic Geometry
- An Review of Chromatic Homotopy Theory
- Derived Moduli Problems and New Cohomology Theories

Spectral Stacks

In derived algebraic geometry , we replace commutative rings with simplicial rings, $\mathbb{E}_\infty$-ring spectra, and so on. One version of derived algebraic geometry is spectral algebraic geometry, which replaces commutative rings with $\mathbb{E}_\infty$-rings.

Definition

A nonconnective spectral Deligne-Mumford stack is a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_X)$ which locally look likes $\mathrm{Sp\acute{e}t} A$, for an E_∞ ring A. We say X is a spectral Deligne-Mumford stack, if all such A is connective.

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1. We say $X = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a n-truncated Deligne-Mumford stack if the structure sheaf $\mathcal{O}_\mathcal{X}$ is n-truncated.
2. We say $X = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a spectral Deligne-Mumford n-stack if $X(R_0)$ is n-truncated for R_0 a commutative ring. A spectral algebraic space is a Deligne-Mumford 0-stack.

Recognition Criterion

Theorem

A spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a nonconnective spectral Deligne-Mumford stack if and only if it satisfying following conditions:

1. The underlying ringed topos $(\mathcal{X}^{\heartsuit}, \pi_0\mathcal{O}_{\mathcal{X}})$ is a classical Deligne-Mumford stack.
2. The canoncial geometric morphism $\phi_* : \mathcal{X} \to \mathrm{Shv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})$ is étale.
3. The homotopy group $\pi_n\mathcal{O}_{\mathcal{X}}$ is a quasi-coherent sheaf on $(\mathcal{X}^{\heartsuit}, \pi_0\mathcal{O}_{\mathcal{X}})$.
4. $\mathcal{O}_{\mathcal{X}}$ is a hypercomplete sheaf.

Spectral Varieties and Spectral p-Divisible Groups

Definition

A spectral variety X over an E_∞-ring R is a nonconnective spectral Deligne-Mumford stack X, such that $\tau_{\geq 0}X \to \mathrm{Spet}\tau_{\geq 0}R$ is flat, proper, locally almost of finite presentation, geometrically reduced and geometrically connected.

- Abelian varieties over R : commutative monoidal objects of $\mathrm{Var}(R)$.
- Spectral elliptic curves over R: spectral abelian varieties of dimension 1 over R.
- Strict elliptic curves over R: abelian group objects of $\mathrm{Var}(R)$ with dimension 1.

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Definition

A height h p-divisible group over an $\mathbb{E}_\infty$-ring A is a functor $X : (\mathrm{Ab}^p_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{FFG}(A)$ with the following conditions:

1. $X(0)$ is trivial.
2. X send exact sequence to fiber sequence.
3. X(M) has degree $|M|^h$ over A for a finite p-group M.

Deformations and Orientations

Let G_0 be a p-divisible group over R_0, a spectral deformation of G_0 along $\rho_A : A \to R_0$ is a pair (G, α), where G is a spectral p-divisible group over A and $\alpha : G_0 \simeq \rho_A^* G$.

Theorem (Lurie '18)

There exists a connective $\mathbb{E}_\infty$-ring $R_{G_0}^{un}$ with a morphism $\rho : R_{G_0}^{un} \to R_0$, such that for other $\rho_A : A \to R_0$, the extension of scalars induces an equivalence of ∞-categories

$$\mathrm{Map}_{\mathrm{CAlg}_{/R_0}}(R_{G_0}^{un}, A) \to \mathrm{Def}_{G_0}(A, \rho_A).$$

- An orientation of an 1-dimensional spectral formal group G over an $\mathbb{E}_\infty$-ring R is a map $e : S^2 \to \Omega^\infty G(\tau_{\geq 0} R)$ which satisfies certain conditions.

Theorem (Lurie '18)

There exists an $\mathbb{E}_\infty$-ring $\mathcal{D}_G$ and $e \in \mathrm{Or}(X_{\mathcal{D}_G})$, such that for other $R' \in \mathrm{CAlg}_R$

$$\mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{D}_G, R') \to \mathrm{Or}(G_{R'}).$$

Elliptic Cohomology

An elliptic cohomology consists of a triple (E, C, ϕ), where E is an even periodic spectrum, C is an elliptic curve C over $\pi_0 E$, $\phi : G_E \cong \hat{C}$ is an isomorphism of group.

Theorem(Goerss-Hopkins-Miller-Lurie)

There is a sheaf $\mathcal{O}_{tmf}$ of E_∞-ring spectra over the stack $\overline{\mathcal{M}}_{ell}$ for the étale topology. For any étale morphism $f : \mathrm{Spec}(R) \to \overline{\mathcal{M}}_{ell}$, there is a natural structure of elliptic spectrum $(\mathcal{O}_{tmf}(f), C_f, \phi)$, satisfying $\pi_0 \mathcal{O}_{tmf}(f) = R$, and C_f is a generalized elliptic curve over R classified by f.

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- There exists a nonconnective spectral Deligne-Mumford stack $\mathcal{M}_{ell}^{or}$ such that

$$\mathrm{Map}_{\mathrm{SpDM}^{nc}}(\mathrm{Spét} R, \mathcal{M}_{ell}^{or}) \cong \mathrm{Ell}^{or}(R)^{\simeq}$$

- The étale topos $\mathcal{U}$ of M_{ell} is the full subcategory of the underlying topos $\mathcal{X}$ of $\mathcal{M}_{ell}^{s}$. We have a map $\phi : \mathcal{M}_{ell}^{or} \to \mathcal{M}_{ell}^{s}$, we consider the direct image sheaf $\phi_* \mathcal{O}_{\mathcal{M}_{ell}^{or}}$, which is sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. We get a functor $\mathcal{O}_{\mathcal{M}_{ell}}^{Top} : \mathcal{U}^{op} \to \mathrm{CAlg}$. This procedure can be viewed as a construction of elliptic cohomology.

Morava E-theories

Let G_0 be a formal group, we have a Morava E-theory $E(n)$, which corresponds the universal deformation of G_0, $\pi_* E(n) = W(k)[\![v_1, \cdots, v_{n-1}]\!][\beta^{\pm 1}]$.

Theorem (Lubin-Tate， 1966)

There is a universal formal group G over $R_{LT} = W(k)[[v_1, \cdots, v_n - 1]]$ in the following sense: for every infinitesimal thickening A of k, there is a bijection

$$\mathrm{Hom}_{/k}(R_{LT}, A) \to \mathrm{Def}(A).$$

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1. Let $\hat{G}_0$ be a formal group over k, it can be viewed as identity component of a connected classical p-divisible group G_0 over k.
2. There exists a universal spectral deformation G over the spectral deformation ring $R^{un}_{G_0}$. Let G^o be the identity component of G, and $R^{or}_{G_0}$ be the orientation classifier of the identity component G^o.
3. $E_{G_0} = L_{K_n} R^{or}_{G_0}$ is even periodic, it satisfies the same properties with Morava E-theories.
4. Uniqueness of Morava E-theories.

Lurie's Theorem

Theorem (Lurie , 2010-2018)

Let M^n_{pd} denote the moduli stack of one dimensional height n p-divisible groups, then there is a sheaf of $\mathbb{E}_\infty$-rings $\mathcal{O}^{\mathrm{Top}}$ on the étale site of M^n_{pd} , such that for any

$$E := \mathcal{O}^{\mathrm{Top}}(\mathrm{Spec} R \xrightarrow{G} M^n_{pd})$$

we have

$$\mathrm{Spf} \pi_0 E^{\mathbb{C}P^\infty} = G^o$$

where G^o is the formal part of the p-divisible group G.

Problems: The universal objects of deformations with level structures are not étale over M^n_{pd}. How do we lift those objects to spectra version?

Derived Relative Cartier Divisors

For a spectral Deligne-Mumford stack X/S, a derived relative Cartier divisor is a morphism $D \to X$ such that $D \to X$ is a closed immersion, the ideal sheaf of D is a line bundle over X, and the morphism $D \to S$ is flat, proper and locally almost of finite presentation.

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Theorem (Ma '24)

Suppose that E is a spectral algebraic space over a connective $\mathbb{E}_\infty$-ring R, such that $E \to R$ is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected, then the functor

$$\begin{aligned} \mathrm{CDiv}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{cn} \to \mathcal{S} \\ & R' \mapsto \mathrm{CDiv}(E_{R'}/R') \end{aligned}$$

is representable by a spectral algebraic space which is locally almost of finite presentation over R.

Derived Level Structures of Spectral Elliptic Curves

For A a finite abelian group, a derived A-level structure of a spectral elliptic curve E/R is a relative Cartier divisor $D \to E$ satisfying its restriction to the heart comes from an ordinary A-level structure.

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Theorem (Ma '24)

For a spectral elliptic curve E over a connective $\mathbb{E}_\infty$-ring R, the functor

$$\begin{aligned} \mathrm{Level}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \\ & R' \mapsto \mathrm{Level}(\mathcal{A}, E_{R'}/R') \end{aligned}$$

is representable by an affine spectral Deligne-Mumford stack.

Derived Level Structures of Spectral p-Divisible Groups

Let G/R be a height h spectral p-divisible group, a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure

$$\phi : D \to G[p^k]$$

of $G[p^k]$. We let $\mathrm{Level}(k, G/R)$ denote the ∞-groupoid of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structures of G/R.

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Theorem (Ma '24)

Suppose G is a spectral p-divisible group of height h over a connective $\mathbb{E}_\infty$-ring R, then the functor

$$\mathrm{Level}^k_{G/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}(k, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S(k) = \mathrm{Sp\acute{e}t}\mathcal{P}^k_{G/R}$.

Representability Theorem

Spectral Artin Representability Theorem (Lurie, 2004-2018)

Let $M : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and R be a Noetherian $\mathbb{E}_\infty$-ring such that $\pi_0 R$ is a Grothendieck ring. Suppose $f : M \to \mathrm{Spec} R$ is a natural transformation and we have

1. $M(R_0)$ is n-truncated for any discrete commutative ring R_0;
2. M is an étale sheaf;
3. M admits a connective cotangent complex L_M;
4. M is nilcomplete, integrable and infinitesimally cohesive;
5. f is locally almost of finite presentation,

then M is representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over R.

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is

- **infinitesimally cohesive**, if for every pull-back diagram on the left in $\mathrm{CAlg}^{\mathrm{cn}}$ such that $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective whose kernel are nilpotent ideals in $\pi_0 A$ and $\pi_0 B'$, the induced diagram is a pull-back square in $\mathcal{S}$.

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \xrightarrow{g} & B \end{array} \qquad\qquad \begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow X(f) \\ X(B') & \xrightarrow{X(g)} & X(B) \end{array}$$

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$$X(R) \to \lim_{\leftarrow n} X(\tau_{\leq n} R)$$

is a homotopy equivalence.

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- **nilcomplete**, if for every connective E_∞-ring R, the canonical map

$$X(R) \to \varprojlim_n X(\tau_{\leq n} R)$$

  is a homotopy equivalence.

- **integrable**, if for a local Noetherian E_∞-ring which is complete with respect to its maximal ideal $m \subset \pi_0 A$, the canonical map

$$X(A) \to \varprojlim_n X(A/m^n)$$

  is a homotopy equivalence.

Moduli Stack of Spectral Elliptic Curves with Derived Level Structures

Theorem (Ma '24)

$$\begin{aligned} \mathcal{M}_{ell}(\mathcal{A}) \quad : \quad & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{ell}(\mathcal{A})(R) = \mathrm{Ell}(\mathcal{A})(R) \end{aligned}$$

is representable by a spectral Deligne-Mumford stack.

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is representable by a spectral Deligne-Mumford stack.

Let $\{R \to U_i\}$ be an étale cover of R, and $U_\bullet$ be the associate check simplicial object. We consider the following diagram

$$\begin{array}{ccc} \mathrm{Ell}(\mathcal{A})(R) & \xrightarrow{f} & \lim_\Delta \mathrm{Ell}(\mathcal{A})(U_\bullet) \\ \downarrow p & & \downarrow q \\ \mathrm{Ell}(R) & \xrightarrow{g} & \lim_\Delta \mathrm{Ell}(U_\bullet). \end{array}$$

p is a left fibration between Kan complexes, so is a Kan fibration. The right vertical map is a pointwise Kan fibration. By picking a suitable model for the homotopy limit, we may assume that q is a Kan fibration as well. We have g is an equivalence .

To prove that f is a equivalence, we only need to prove that for every $E \in \mathrm{Ell}(R)$, the map

$$p^{-1}E \simeq \mathrm{Level}(\mathcal{A}, E/R) \to \lim_{\Delta} \mathrm{Level}(\mathcal{A}, E \times_R U_\bullet / U_\bullet) \simeq q^{-1}g(E)$$

is an equivalence. But this is true due to étaleness of derived level structures.

Spectral Deformations with Derived Level Structures

Suppose G_0 is a p-divisible group of height h over a perfect F_p-algebra R_0. We consider the following functor

$$\begin{aligned} \mathcal{M}_k^{\mathrm{or}} \quad : \quad & \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{\mathrm{or}}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, k)$ is the ∞-category spanned by those quadruples (G, ρ, e, η)

1. G is a spectral p-divisible group over R.
2. ρ is an equivalence class of G_0-taggings of R.
3. e is an orientation of the identity component of G.
4. $\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G/R.

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Theorem (Ma '24)

The functor $\mathcal{M}_k^{or}$ is corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JL}_k$, where $\mathcal{JL}_k$ is a finite $R_{G_0}^{or}$-algebra, $R_{G_0}^{or}$ is the orientation deformation ring of G_0.

We call this spectrum $\mathcal{JL}_k$ the Jacquet-Langlands spectrum. It is easy to see that this $\mathcal{JL}_k$ admit an action of $GL_h(\mathbb{Z}/p^k\mathbb{Z}) \times \mathbb{G}_n$. When k varies, we have a tower

$$\begin{array}{c} \cdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\mathcal{JL}_k \\ \downarrow \\ \mathrm{Sp\acute{e}t}\mathcal{JL}_{k-1} \\ \downarrow \\ \cdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\mathcal{JL}_0. \end{array}$$

We call this tower the higher categorical Lubin-Tate tower.

The Langlands duals of Morava E-theories

- $\mathcal{J}\mathcal{L} = \lim\limits_{\leftarrow} \mathcal{J}\mathcal{L}_k$
- We have actions $\mathbb{G}_n \times \mathrm{GL}_n(\mathbb{Z}_p) \curvearrowright \mathcal{J}\mathcal{L}$.

The Langlands duals of Morava E-theories

- $\mathcal{JL} = \varprojlim \mathcal{JL}_k$
- We have actions $\mathbb{G}_n \times \mathrm{GL}_n(\mathbb{Z}_p) \curvearrowright \mathcal{JL}$.
- We define the dual Morava E-theory ${}^L E_n$ to be $\mathcal{JL}^{hG_n}$.
- We have convergent spectral sequences

$$E_2^{s,t} \cong H^s_{cts}(\mathbb{G}_n \times GL_n(\mathbb{Z}_p), \pi_t \mathcal{JL}) \Longrightarrow \pi_{t-s} L_{K(n)} S^0.$$

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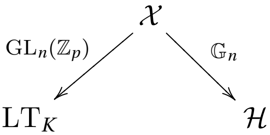
Theorem(Barthel-Schlank-Stapleton-Weinstein， 2024)

There is an isomorphism of graded $\mathbb{Q}$-algebras

$$\mathbb{Q} \otimes \pi_* L_{K(n)} S^0 \cong \Lambda_{\mathbb{Q}_p}(\zeta_1, \zeta_2, \cdots \zeta_n),$$

where the latter is the exterior $\mathbb{Q}_p$-algebra with generators ζ_i in degree $1 - 2i$.

They use the following diagram in their proof.



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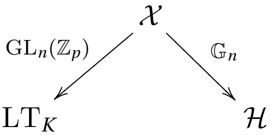
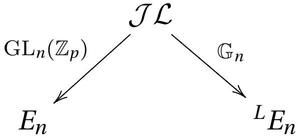


It can be lift to the following diagram in the level of spectra.



Thanks for Listening !