Research Statement

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My research interests are Algebraic Topology, Algebraic Geometry, Number Theory and Theoretical Physics. I focus on moduli spaces and Langlands duality in these fields.

Contents

1 Derived Moduli Spaces

In [Lur09] and [Lur18], Lurie uses spectral algebraic geometry methods give a proof of Goerss-Hopkins-Miller theorem for topological modular forms. Except the application of elliptic cohomology, Lurie also proved the E_∞ structures of Morava E-theories [Lur18], which use the spectral version of deformations of formal groups and p-divisible groups. There are its rising applications in algebraic topology. Like topological automorphic forms [BL10], the construction of equivariant topological modular forms [GM20], elliptic Hochschild homology [ST23] and so on.

The moduli problems of deformations of formal groups with level structures are also representable and moduli spaces of different levels form a Lubin-Tate tower [RZ96], [FGL08]. We know that the universal objects of deformations of formal groups have higher algebra analogues which are Morava E-theories. A natural question is what are the higher categorical analogues of the moduli problems of deformations with level structures? And can we find higher categorical analogues of the Lubin-Tate towers. Unfortunately, the representable object of deformations with level structures is not étale over the universal deformations, so we can't use the Goerss-Hopkins-Miller theorem directly. Except this, in the computation of unstable homotopy groups of spheres, after applying the EHP-spectral sequences and Bousfield-Kuhn functor, we find some terms in E_2-page also comes form the universal deformation of isogenies of form groups. They are computed by the Morava E-theories on the classify spaces of symmetric groups [Str97], [Str98]. They can be viewed as sheaves on the Lubin-Tate tower. We hope a more conceptual view about this fact in the higher categorical Lubin-Tate tower.

Question 1.1 What are the higher categorical analogues of level structures? Can they form good derived moduli spaces?

In the upcoming paper [Ma24], we give some attempts on this problem, we define the derived level structures in the context of spectral algebraic geometry and give some representable results about moduli problems associated with derived level structures, at least in the case of spectral elliptic curves.

For representability reason, we use moduli associated with sheaves to detect higher homotopy of derived versions of level structures. For a spectral Deligne-Mumford stack X/S, a relative Cartier divisor is a morphism $D \to X$ of spectral Deligne-Mumford stacks such that $D \to X$ is a closed immersion, the morphism $D \to S$ is flat, proper and locally almost of finite presentation and the ideal sheaf of D is a line bundle over X. Then we use Lurie's representability theorem prove that the relative Cartier divisor is representable by a spectral Deligne-Mumford stack.

Theorem 1.2. *[Ma24] Let E/R be a spectral algebraic space which is flat, proper, locally almost of finite presentation and geometric connected, then the functor*

$$\begin{aligned} \mathrm{CDiv}_{E/R} \quad : \quad & \mathrm{CAlg} \to \mathcal{S} \\ & R' \mapsto \mathrm{CDiv}(E_{R'}/R') \end{aligned}$$

is representable by a spectral algebraic space which is locally almost of finite presentation over R.

We define derived level structures of spectral elliptic curves. For A an abstract abelian group, which represents the classical $\Gamma(N), \Gamma_0(N)$ level structures of elliptic curves. Roughly speaking, a derived A-level structure of a spectral elliptic curve E over an $\mathbb{E}_\infty$-ring R is just a relative Cartier divisor $D \to E$ satisfying its restriction to the heart comes from an ordinary A-level structure. We let $\mathrm{Level}(A, E/R)$ denote the space of derived A-level structures of E/R. We prove that moduli problems associated with derived level structures are representable.

Theorem 1.3. *[Ma24] Let E/R be a spectral elliptic curve, then the functor*

$$\begin{aligned} \mathrm{Level}_{E/R} \quad : \quad & \mathrm{CAlg}^{cn} \to \mathcal{S} \\ & R' \mapsto \mathrm{Level}(A, E_{R'}/R') \end{aligned}$$

is representable by an affine spectral Deligne-Mumford stack which is locally almost of finite presentation over R.

For the application of derived level structures, we first prove that the moduli stack of spectral elliptic curves with derived level structures is representable by a spectral Deligne-Mumford stack. We let $\mathrm{Ell}(A)(R)$ denote the space of spectral elliptic curves with derived A-level structures over a connective $\mathbb{E}_\infty$-ring R.

Theorem 1.4. *[Ma24] The functor*

$$\begin{aligned} \mathcal{M}_{ell}(A) \quad : \quad & \mathrm{CAlg}^{cn} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{ell}(A)(R) = \mathrm{Ell}(A)(R) \end{aligned}$$

is representable by a spectral Deligne-Mumford stack, which is locally almost of finite presentation over the sphere spectrum $\mathbb{S}$.

As we said, what we want are the higher categoric analogues of Lubin-Tate towers, so we need to consider the moduli problem of spectral derived deformations with derived level structures.

Let G_0 be a height h p-divisible group over a commutative ring R_{G_0}. We consider the following functor:

$$\begin{aligned} \mathcal{M}^{or}_m \quad : \quad & \mathrm{CAlg}^{cn}_k \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{or}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, k)$ is the ∞-category whose objects are four-tupes (G, ρ, e, η)

1. G is a spectral p-divisible group over R.
2. ρ is an equivalence class of G_0 taggings of R.
3. e is an orientation of the identity component of G.
4. $\eta : D \to G$ is a derived $(\mathbb{Z}/\mathbb{Z}^k)^h$ level structure.

Theorem 1.5. *[Ma24] The functor $\mathcal{M}^{or}_k$ is representable by a formal spectral Deligne-Mumford stack* $\mathrm{Spf}\mathcal{JL}_k$*, where $\mathcal{JL}_k$ is an E_∞-ring which is finitely generated over $R^{or}_{G_0}$, here $R^{or}_{G_0}$ is the orientated deformation ring defined in [Lur18]. We call this spectrum Jacquet-Langlands spectrum.*

Question 1.6 What properties does these spectra $\mathcal{JL}_k$ have?

It is easy to see that these $\mathcal{JL}_k$ admit an action of $GL_n(Z/p^k Z) \times \mathrm{Aut}(G_0)$, which can be viewed as a higher analogue of Lubin-Tate tower. Passing to the limit of m, we get action of $GL_n(\mathbb{Z}_p)$ on certain spectra. And when k varies, we have a tower

$$\begin{array}{c} \cdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\mathcal{JL}_k \\ \downarrow \\ \mathrm{Sp\acute{e}t}\mathcal{JL}_{k-1} \\ \downarrow \\ \cdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\mathcal{JL}_0. \end{array}$$

We call this tower higher categorical Lubin-Tate tower. This a tower of spectral stacks and π_0 of this tower is the classical Lubin-Tate tower. We hope find more properties of this tower in later study.

It follows that [BMS19], some topological realizations of classical cohomology rings may have good structures, like the topological Hochschild homology of quasiregualr semiperfectoid rings. These leads to the establishment of some special p-adic cohomology theories, Breuil-Kisin cohomology theory and its refinement, prismatic cohomology. The heart of this topic is δ-rings and its topological realization derived δ-rings [Hol23].

Question 1.7 Does these derived rings follows form derived moduli problems?

Actually, in [Lur18], Lurie constructed the spherical Witt-vectors, which follows form derived moduli problems, thickenings of relatively perfect morphisms . And it has many application in chromatic homotopy theory, like [BSY22] and [Ant23]. We hope to establish more derived moduli problems, to give us more understand of these derived rings. And we hope the spectrum $\mathcal{JL}_m$ is a good p-adic cohomology theory. It will give us more arithmetic information.

Question 1.8 Computation of $\mathcal{JL}_m$ theory of some p-adic rings, especially for perfectoid rings.

We consider the spherical Witt-vector functor defined in [Lur18] and [BSY22].

$$\mathbb{SW} : \mathrm{Perf} \to \mathrm{CAlg}(\mathrm{Sp}_p).$$

By this functor and the classical algebraic methods, like power operations, periodictiy, thick subcategories, it may give us more information about the ∞-category of derived δ-rings. The reason we consider chromatic methods is the appearance of v_1 periodic elements in the computation of topological Hochschild homology. We hope through the study of global properties of derived δ-rings, we can find more computation methods of K-theory and its local variant of perfectoid rings.

Question 1.9 The structures and classification of derived δ-rings.

2 Representation Theory in Derived Algebraic Geometry

We know actions of certain Galois groups and automorphism groups on certain objects, like Morava E-theories, THH, TC. And this means that these groups acting on their homotopy groups. But can these actions lift to derived category, i.e., we want to find representations in derived category. And there are other reasons we need to do representation in derived category. For example, let $\mathbb{E}_n$ a formal group og height n over a perfect field k with char k=p, we have a spectral sequence

$$E_2^{s,t} \simeq H^s_{cts}(\mathbb{G}_n, \pi_t E_n) \Longrightarrow \pi_{t-s} L_{K(n)} S^0$$

where E_n is the associated Morava E-theory and $\mathbb{G}_n$ is the Morava stabilizer group. So when consider resolutions of Morava E-theories, it is reasonable ot consider Sp_H, the category of spectra admits H actions, where H is a subgroup of $\mathbb{G}_n$ with finite index.

We hope to establish representation theory in derived category, like $D(R)$, $D(\mathrm{QCoh}(X))$, Sp, dStack. But in general the derived category of G-objects in $\mathrm{Mod}(R)$ is not equal to the category of G-objects in $D(R)$. In algebraic topology, it seems that group actions of spectra are more easy to find, like actions of Morava stabilizer groups on Morava E-theories. We proposed an viewpoint that how do we use spectral algebraic geometry to solve this problem.

1. Representations in $\mathrm{Var}_k, \mathrm{QCoh}(X)$;
2. Explain these $\mathrm{Var}_k, \mathrm{QCoh}(X)$ as classical moduli spaces;
3. Find associated derived moduli problems in spectral algebraic geometry ;
4. Using repersentability theorem to get derived geometric objects;
5. Representations in derived categories.

Now, let's see some examples of this strategy.

Example 2.1 (**Spherical Witt Vectors**) We consider the spherical Witt-vector functor defined in [Lur18] and [BSY22].

$$\mathbb{SW} : \mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{CAlg}(\mathrm{Sp}_p).$$

form the category of perfect $\mathbb{F}_p$ algebras to the ∞-category of p-complete $\mathbb{E}_\infty$-rings. This functor is defined by studying a derived moduli problem, thickenings of relatively perfect morphisms. And it has many application in chromatic homotopy theory, like [BSY22] and [Ant23]. And it is easy to see that this functor can find some Galois representations in derived category.

Example 2.2 (**Spectral Deformations of p-Divisible Groups**) For a classical p-divisible group G_0 over a perfect field k, we consider the Morava stabilizer group $S = \mathrm{Aut}(G_0) \rtimes Gal(k)$. We can consider its spectral deformations over an $\mathbb{E}_\infty$-ring R, which consists of pairs (G, ρ), where G is a spectral p-divisible group over R, and ρ is an equivalence class of G_0 taggings. In [Lur18], Lurie proved that there exits an universal deformation of G_0. i.e., there exists a complete adic $\mathbb{E}_\infty$-ring $R^{un}_{G_0}$, and a morphism $\rho : R^{un}_{G_0} \to R_0$ such that the functor Def_{G_0} is corepresentable by $R^{un}_{G_0}$. i.e., for any complete adic $\mathbb{E}_\infty$-ring R, there is an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R^{un}_{G_0}, R) \to \mathrm{Def}_{G_0}(R).$$

It is easy to see that this spectrum $R^{un}_{G_0}$ admits an action of S.

Example 2.3 (**Derived Level Structures**) Let k be a p-adic field with residue field k of characteristic p. Let LT_n denote the moduli space of deformations with level $(\mathbb{Z}/\mathbb{Z}^n)^h$-structures of a height h formal group G_0 . Passing to the direct limit over n of vanishing cycle sheaves of LT_n. This give an collection $\{\Psi^i_m\}$ of infinite-dimensional $\bar{\mathbf{Q}}_l$-vector spaces which admits admissible nature actions of the subgroup of $GL_g(K) \times D^\times_{K,g} \times W_K$. By our construction of derived level structures, we find these actions can lift to actions on certain ∞-spectra.

We want to develop a general representation theory in E_∞-spectra, spectral schemes, and spectral stacks, such that it is compatible with the classical definition of actions of algebraic groups on schemes.

Let G be an algebraic group, viewed as a 0-truncated spectral Deligne-Mumford stack, Let X be a spectral Deligne-Mumford stack admits a G-action. Then does this make $\Gamma(X, \mathcal{O}_X)$ to become a G-equivariant spectrum? See [MLC+96] for equivariant spectra and [BH15] for the equivariant $\mathbb{E}_\infty$-rings setting. On the other hand, what is the meaning of the action of an algebraic group on a spectrum, since spectra are topological, they don't have algebraic structures.

Question 2.4 What is the right definition of algebraic groups acting on spectra stacks?

The reason we asked this question is that when we say a spectrum admits a G-action. If G is topological group, it is fine. But when G is a algebraic group, how do we say this action is compatible with the algebraic structure?

Question 2.5 Does the principal bundle theory holds in spectral algebraic geometry? What good properties does spectral version Bun_G have in spectral algebraic geometry?

Just like the classical algebraic geometry, we want the moduli stack of principal bundles over a curve has a good structure, but there are still something unknow in spectral algebraic geometry.

The study of Bun_G is related with many topics in algebraic geometry, number theory and mathematical physics. D-modules are important description for moduli spaces, which is useful in geometric representation theory and Langlands correspondence. To define D-modules, we need to study differential operators in spectral algebraic geometry. Infinitesimal deformations over a field k of characteristic zero are governed by differential graded Lie algebras. This paradigm, which was formalised by Lurie [LD11] and Pridham [Pri10], was recently generalized to arbitrary fields [BM19]. Over $\mathbb{E}_\infty$-rings, formal moduli are equivalent to spectral partition Lie algebras. These are chain complexes with extra structure, which is parameterized by a sifted-colimit-preserving monad monad $\mathrm{Lie}^\pi_{k,\Delta}$ is defined by the following properties

1. If V is a finite dimensional k-vector space, then $\mathrm{Lie}^\pi_{k,\Delta}(V)$ is the linear dual of the algebraic cotangent fiber of $k \oplus V^\vee$, the trivial square-zero extension of k by $V^\vee$.

2. If $V \simeq \mathrm{Tot}(V^\bullet)$ is represented by a cosimplicial k-vector space $V^\bullet$, then

$$\mathrm{Lie}^\pi_{k,\Delta}(V) = \bigoplus_n \mathrm{Tot}(\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes (V^\bullet)^{\otimes n})^{\Sigma_n}.$$

Here $\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k)$ denote the k-valued cosimplices on the space $\Sigma|\Pi_n|^\diamond$, the functor $(-)^{\Sigma_n}$ takes the strict fixed points, and the tensor product is computed in cosimplicial k-modules.

3. The functor $\mathrm{Lie}^\pi_{k,\Delta}$ commuted with filtered colimits and geometric realizations.

4. The tangent fiber T_X of any $X \in \mathrm{Moduli}_{k,\Delta}$ has the structure of a $\mathrm{Lie}^\pi_{k,\Delta}$-algebra.

Theorem 2.6. *([BM19]) If k is a field, there is an equivalence of ∞-categories*

$$\mathrm{Moduli}_{k,\Delta} \simeq \mathrm{Alg}_{\mathrm{Lie}^\pi_{k,\Delta}}$$

between formal moduli problems and partition Lie algebra k. It sends a formal moduli problem $X \in \mathrm{Moduli}_{k,\Delta}$ *to its tangent fibre* T_X *equipped with a suitable partition Lie algebra structure.*

Definition 2.7 For a spectral Deligne-Mumford stack X, we let D_X denote the ∞-category $\mathrm{Lie}^\pi_{k,\Delta}$-algebra objects ϕ in $\mathrm{End}(\mathcal{O}_X)$ satisfying for any étale morphism $i : \mathrm{Sp\acute{e}t}R \to X$ and any two sections $s, t : R \to {}^*\mathcal{O}_X$, we have a equation

$$\phi|_{\mathrm{Sp\acute{e}t}R}(st) = s\phi|_{\mathrm{Sp\acute{e}t}R}(t) + \phi|_{\mathrm{Sp\acute{e}t}R}(s)t$$

Remark 2.8 This is just a naive definition, the reason we used spectral partial Lie algebras, is we want use the Koszul duality in spectra.

Question 2.9 For a spectral Deligne-Mumford stack X, is the category D_X-Mod equivalent to $\mathrm{QCoh}(X_{dR})$?

3 Topological Langlands Correspondences

Let E be a local field, G be a reductive group over E. The classical local Langlands correspondence predict that for any irreducible smooth representation π of $G(E)$, we can naturally associate an L-parameter

$$\phi_E : W_E \to G(\mathbb{C}).$$

In the classical arithmetic geometry, the Lubin-Tate tower can be used to realize the Jacquet-Langlands correspondence [HT01]. Is there a topological realization of the Jacquet-Langlands correspondence? Actually, in a recent paper [SS23], they already realized a version of topological Jacquet-Langlands correspondence. But their method is based on the Goerss-Hopkins-Miller-Lurie sheaf. They actually consider the degenerate level structures such that representing object is étale over representing object of universal deformations.

We hope our higher categorical analogues of Lubin-Tate towers can also establish a topological version of the classical Langlands correspondence, which means that we construct representations on the category of spectra. Our derived level structure give an attempt on this idea by considering certain function spectra. Let $\mathbb{G}$ be a formal group over a field of characteristic p. By the construction of Jacquet-Langlands spectra above, it is easy to see that this $\mathcal{JL}_m$ admit an action of $GL_h(Z/p^mZ) \times \mathrm{Aut}(G_0)$. Let $\mathcal{JL}$ be its ℓ-adic complete Jacquet-Langlands spectrum. and X be a spectrum with an action of $\mathrm{Aut}(\mathbb{G}_n)$. We have the following conjecture.

Conjecture 3.1. *The function spectrum* $F(X, \mathcal{JL})$ *admits an action of* $GL_n(\mathbb{Z}_p)$ *and all its homotopy groups are* $\mathbb{Z}_l$*-modules.*

We know actions of certain Galois groups and automorphism groups on certain objects, like Morava E-theories, THH, TC. And this means that these groups acting on their homotopy groups. By the

Langlangs correspondence, we can associated certain objects which have the action of GL_n, or more generally, reductive groups. But can these objects lift to GL_n equivalent spectra.

Generally we want to know how does actions of Galois side on certain objects can related to actions of some algebraic groups on another certain objects. And the name topological Langlands correspondence comes from that we want certain spectral algebraic geometry objects play the roles of homotopy representations of Langlands dual groups, which can be viewed as automorphic side of topological Langlands correspondence.

Question 3.2 Find a topological refinement of arithmetic Langlands correspondence.

In [?], Galatius and Venkatesh define and study derived Galois deformations. Let F be a global field, S is a finite set of places of F. Let k be a finite filed, G be a split algebraic group over the Witt vectors $W(k)$. Let $\bar{\rho}$ be a representation of $\pi_1 \mathrm{Spec}\mathcal{O}_F[1/s]$ in $G(k)$. Then we can define the Galois deformation functor $M^{\bar{\rho}}_{\mathcal{O}_F[1/s]}$ form the category of Artinian local $W(k)$-algebras augmented over k to the category of sets, by send A to the set of diagrams of the form

$$\begin{array}{ccc} & & G(A) \\ & \overset{\rho}{\nearrow} & \downarrow \\ \pi_1 \mathrm{Spec}\mathcal{O}_F[1/s] & \xrightarrow{\bar{\rho}} & G(k) \end{array}$$

modulo conjugacy. We notice that the étale homotopy type of the scheme $\mathrm{Spec}\mathcal{O}_F[1/s]$ is equal to the classify space of $\pi_1 \mathrm{Spec}\mathcal{O}_F[1/s]$. After applying the classifying space functor, and notice that G can be extend to simplicial rings. We then define the derived Galois deformation functor form the category of Artinian simplicial rings to the category of spaces by sending a simplicial ring $\mathcal{A}$ to diagrams

$$\begin{array}{ccc} & & BG(\mathcal{A}) \\ & \overset{\rho}{\nearrow} & \downarrow \\ \acute{E}t(\mathcal{O}_F[1/s]) & \xrightarrow{\bar{\rho}} & BG(k). \end{array}$$

It can be prove that this derived moduli problem is representable by a simplicial ring $\mathcal{R}^{\bar{\rho}}_{\mathcal{P}_F[1/S]}$, and its π_0 is the classical Galois deformation ring. There are variants of this construction, such as local derived deformation functor and crystalline deformation functor, see [?] for more details.

Let G be a reductive group over a local filed K, $U \subset G$ be a compact open subgroup. Let A be a commutative ring, we let $A[G(K)/U] = c - \mathrm{Ind}_U^{G(K)} A$ denote the induced representation of the trivial representation from U to $G(K)$. The classical Hecke algebra for the pair $(G(K), U)$ is

$$H(G(K), U : A) := \mathrm{Hom}_{G(K)}(A[G(K)/U], A[G(K)/U]).$$

In [?], Venkatesh define the derived Hecke algebra to be

$$\mathcal{H}(G(K), U; A) := \mathrm{Ext}^*_{G(K)}(A[G(K)/U], A[G(K)/U]).$$

It satisfies certain good properties like the classical Hecke algebra.

These two construction give us evidence about homotopical version of Langland correspondence for general reductive group G, but the derived Hecke algebra doesn't comes from symmetry of derived objects

Question 3.3 Find certain derived objects whose symmetry can be described by derived Hecke algebra.

In recent paper [?] and [?], there are some construction of Hecke operation on topological modular forms. We hope to establish a general theory of Hecke algebra in the derived algebra geometry context. In the geometric Langlands correspondence, the construction of Hecke stack is an important ingredient. We want to find a reasonable construction of derived Hecke stack which is compatible for Hecke algebra of topological modular forms.

4 Two-Dimensional Langlands Correspondences

We know that for a reductive group G, and a global field F, the arithmetic Langlands correspondence predict an equivalence between the following two categories:

1. Representations of $\mathrm{Gal}(\bar{F}/F)$, i.e., morphisms $\mathrm{Gal}(\bar{F}/F) \to {}^LG$;
2. Automorphism representations of $G(\mathbb{A}_F)$.

For $G = GL_1$, this correspondence is actually the global class filed theory. It is realized by the global Artin reciprocity map

$$\Psi_{L/K} : C_K \to \mathrm{Gal}(L/K)^{ab},$$

here L/K is a finite global field extension. The local Langlands correspondence for GL_1 is just local class filed theory. It is realized by the local Artin reciprocity map:

$$\Psi_{L/K} : K^* \to \mathrm{Gal}(L/K)^{ab},$$

here L/K is a finite local field extension.

The geometric Langlands correspondence actually aim to construct an equivalence of categories

$$D(\mathrm{QCoh}(\mathrm{LocSys}_{^LG}(X)) \simeq D(\mathcal{D}(\mathrm{Bun}_G)),$$

from the derived category of quasi-coherent sheaves on ${}^LG(X)$ local systems on X and the derived categories of D-modules on the moduli stack of G-bundles over X [BD91]. Due to the work of Fargues-Scholze [FS21], the arithmetic local Langlands correspondence can also be some kinds of geometric Langlands correspondence, but in the perfectoid world.

Classes field theory has a generalization for higher dimensional local field.

Definition 4.1 A 0-dimensional local field is a finite field. For $n \geq 1$, a n-dimensional local field is a field which is complete with respect to a discrete valuation and whose residue field is an $(n-1)$-dimensional local field.

Theorem 4.2. *([Kat77]) If K is a n-dimensional local field, then there exists a natural reciprocity map*

$$K_n^M(k) \to \mathrm{Gal}(\bar{K}/K)^{ab}$$

and for any finite Galois extension l/k, the reciprocity map induces an isomorphism

$$K_n^M(k)/\mathrm{Norm}_{l/k}(K_n^M(l)) \to \mathrm{Gal}(\bar{K}/K)^{ab},$$

where $K_n^M(k)$ is the Milnor K-groups of k.

Theorem 4.3. *([KS86]) Let $\bar{X}$ be a normal connected scheme, projective and of finite type over* $\mathrm{Spec}(\mathbb{Z})$ *and let $X \subseteq \bar{X}$ be a non-empty open regular subscheme. Let $d = \dim(X)$ and assume (for simplicity)* *that $X(\mathbb{R}) = \emptyset$. Then there exists a natural reciprocity map*

$$rec : \varprojlim_{\mathcal{I} \subset \bar{X}, \mathcal{I}|_X = \mathcal{O}_X} H^d_{c.d.}(\bar{X}, \mathcal{K}^M_d(\mathcal{O}_{\bar{X}}, \mathrm{I})) \to \pi_1^{ab}(X)$$

If X is flat over $\mathbb{Z}$, then rec is an isomorphism. If X is a variety over a finite field, then rec is injective and $coker(rec) \cong \hat{\mathbb{Z}}/\mathbb{Z}$.

And there are other descriptions such as relations of Chow groups and abelian étale fundamental groups. Let X be a regular, connected and projective scheme over $\mathbb{Z}$. Assume (for simplicity) that $X(\mathbb{R}) = \emptyset$. Then there is a reciprocity map

$$rec : \mathrm{CH}_0(X) \to \pi_1^{ab}(X).$$

We know that Langlands correspondence is a generalization of class filed theory to general reductive groups. As we have higher class field theory, it is natural to consider higher Langlands correspondence. For a smooth projective curves X, the classical Langlands correspondence predicts for a morphism

$$\pi_1(X) \to {}^L G,$$

there is a function on

$$G(F) \setminus G(A_F)/G(\mathcal{O}_F)$$

where F is the function filed of X, and $\mathcal{O}_F = \prod_{x \in X} \mathcal{O}_{X,x}$. But this double coset is equivalent to the set of isomorphism classes of G-bundles on X.

Let us consider the two dimensional case, assume S is an arithmetic scheme. We then have the function filed F_S, and we can construct two dimensional adeles A_S [Fes08].

Question 4.4 What are 2-dimensional analogs of $G(F) \setminus G(A_F)/G(\mathcal{O}_F)$ and Bun_G in the 2 dimensional Langlands correspondence.

By the higher class filed theory, we can't directly replace the adeles of curves by adeles of surfaces. We can't consider the algebraic stack Bun_G directly. For an algebraic curve X, the moduli stack Bun_G of G-torsors is a algebraic stack. We need to find a suitable moduli stack to describe the compactible G-adeles on the arithmetic scheme S.

Question 4.5 What are the analogs of automorphic functions in 2-dimensional Langlands correspondence?

The 1-dimensional Langlands correspondence predicts certain automorphic representations of $G(\mathbb{A}_A)$. It's natural to consider automorphic functions in 2-dimensional Langlands correspondence. Similar to 2-dimensional case, we can also consider the Hecke action.

Question 4.6 Find the corresponding L-function analogues in 2-dimensional Langlands correspondence.

The 1-dimensional Langlands correspondence also predicts that certain L-functions are related. So we can consider its two dimensional analogues. But to do this, we need to complete the theory of zeta integrals on 2-dimensional adeles, see more details on [Fes08]. And we need the representation theory of reductive groups on automorphic functions of 2-dimensional objects.

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