# Methods of Spectral Algebraic Geometry in Chromatic Homotopy Theory

Doctoral Dissertation Proposal

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1. Introduction to Chromatic Homotopy Theory

2. Introduction to Spectral Algebraic Geometry

3. Application of Spectral Algebraic Geometry

4. How to Lift a Complex Orientation  $MU \rightarrow E$  to an  $E_{\infty}$  Map

# Introduction to Chromatic Homotopy Theory

Generalized cohomology theories of  $\mathsf{Top} \longleftrightarrow \mathsf{Spectra}$ 

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### Formal Groups

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$$\mathcal{O}_G \to \mathcal{O}_{G \times G} \cong \mathcal{O}_G \otimes \mathcal{O}_G$$

 $\mathcal{O}_G$  is just R[X] and  $\mathcal{O}_G \otimes \mathcal{O}_G$  is  $R[X] \otimes_R R[Y] = R[X, Y]$ .

$$\phi : R[X] \to R[X, Y] X \to f(X, Y)$$

## Formal Group Laws

### Definition

Formal group law :  $F \in R[[x_1, x_2]]$ 

- F(x, 0) = F(0, x) = x (Identity)
- $F(x_1, x_2) = F(x_2, x_1)$  (Commutativity)
- $F(F(x_1, x_2), x_3) = F(x_1, F(x_2, x_3))$  (Associativity)

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There exists a ring L and  $F_{univ}(x, y) \in L[x, y]$ 

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#### Lazard's Theorem

 $L\cong\mathbb{Z}[t_1,t_2,\cdots]$ 

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# Heights of Formal Groups

Let  $f(x, y) \in R\llbracket x, y \rrbracket$ 

1. If n = 0, we set [n](t) = 0.

2. If n > 0, we set [n](t) = f([n - 1](t), t).

P-series p[t] is either 0 or equals  $\lambda t^{p^n} + O(t^{p^n+1})$  for some n > 0.

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Let  $v_n$  denote th coefficient of  $t^{p^n}$  in the p-series, f has height  $\leq n$  if  $v_i = 0$  fro i < n, f has height exactly n if it has height  $\leq n$  and  $v_n$  is invertible.

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#### Examples

1. Formal multiplicative group f(x, y) = x + y + xy,  $[n](t) = (1 + t)^n - 1$ . If p = 0 in R, then  $[p](t) = (1 + t)^p - 1 = t^p$ , so f has height 1. Let  $f(x, y) \in R\llbracket x, y \rrbracket$ 

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- Formal additive group f(x, y) = x + y, if p = 0 in R. Then
   [p](t) = 0, so f has infinite height.

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# **Complex Oriented Cohomology Theories**

#### **Complex Orientation**

Let E be cohomology theory. Then a complex orientation of E is a choice  $x \in E^2(\mathbb{C}P^{\infty})$  which restricts to 1 under the composite

$$E^2(\mathbb{C}P^\infty) \to E^2(\mathbb{C}P^1) = E^2(S^2) \cong E^0(*)$$

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 $E^{*}(\mathbb{CP}^{\infty}) \cong E^{*}(*)\llbracket t \rrbracket = (\pi_{*}E)\llbracket t \rrbracket$  $(\pi_{*}E)\llbracket t \rrbracket \cong E^{*}(\mathbb{CP}^{\infty}) \to E^{*}(\mathbb{CP}^{\infty} \times \mathbb{CP}^{\infty}) \cong (\pi_{*}E)\llbracket [X, y] \rrbracket$ {complex oriented cohomology theory  $E \rbrace \to G_{E} = \operatorname{Spf} E^{0}(\mathbb{CP}^{\infty}).$ 

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 $E^*(\mathbb{CP}^{\infty}) \cong E^*(*)\llbracket t \rrbracket = (\pi_* E)[[t]]$  $(\pi_* E)[[t]] \cong E^*(\mathbb{CP}^{\infty}) \to E^*(\mathbb{CP}^{\infty} \times \mathbb{CP}^{\infty}) \cong (\pi_* E)[[X, y]]$  $\{\text{complex oriented cohomology theory} E\} \to G_F = \operatorname{Spf} E^0(\mathbb{CP}^{\infty}).$ 

#### Theorem(Quillen, 1969)

MU is the universal complex oriented cohomology theory,  $L \cong \pi_* MU$ .

For E complex oriented,  $MU \rightarrow E$ , induce  $L = \pi_*MU \rightarrow \pi_*E$ .

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#### Landweber's Exact Functor Theorem, 1976

Let *M* be a module over the Lazard ring L. Then *M* is flat over  $\mathcal{M}_{FG}$  if and only if for every prime number *p*, the elements  $v_0 = p, v_1, v_2, \dots \in L$  form a regular sequence for *M*. **Deformation of formal groups:** Let  $G_0$  be a formal group over a perfect field k with characteristic p, then a deformation of  $G_0$  to R is a triple  $(G, i, \Psi)$  satisfying

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- There is a map  $i: k \to R/m$ ,
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#### Lubin-Tate's Theorem, 1966

There is a universal formal group G over  $R_{LT} = W(k)[[v_1, \dots, v_n - 1]]$ in the following sense: for every infinitesimal thickening A of k, there is a bijection

$$\operatorname{Hom}_{/k}(R_{LT}, A) \to \operatorname{Def}(A).$$

## Morava E-theories and Morava K-theories

Using Landweber exact functor theorem, there is a even periodic spectrum E(n)

$$\pi_* E(n) = W(k) [v_1, \cdots, v_{n-1}] [\beta^{\pm 1}]$$

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M(k) denote the cofiber of the map  $\sum^{2k} MU_{(p)} \rightarrow MU_{(p)}$  given by the multiplication by  $t_k$ .

Let K(n) denote the smash product

$$MU_{(p)}[v_n^{-1}] \otimes_{MU_{(p)}} \bigotimes_{k \neq p^n - 1} M(k).$$

This spectrum *K*(*n*) is called **Morava K-theory**. The homotopy groups of *K*(*n*) is

$$\pi_* K(n) \cong (\pi_* M U_{(p)})[v_n^{-1}]/(t_0, t_1, \cdots t_{p^n-2}, t_{p^n}, \cdots) \cong \mathbb{F}_p[v_n^{\pm 1}]$$

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  - $L_{K(n)}$ , behaves like completion along the locally closed substack  $\mathcal{M}_{FG}^n \subset \mathcal{M}_{FG} \times \operatorname{Spec}\mathbb{Z}_{(p)}$ .

# Elliptic Cohomology

An elliptic cohomology consists of

- 1. An even periodic spectrum E .
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- 3.  $\phi: G_E \cong \hat{C}$

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## Theorem(Goerss-Hopkins-Miller-Lurie)

There is a sheaf  $\mathcal{O}_{tmf}$  of  $E_{\infty}$ -ring spectra over the stack  $\overline{\mathcal{M}}_{ell}$  for the *étale* topology. For any *étale* morphism  $f: \operatorname{Spec}(R) \to \overline{\mathcal{M}}_{ell}$ , there is a natural structure of elliptic spectrum  $(\mathcal{O}_{tmf}(f), C_f, \phi)$ , satisfying  $\pi_0 \mathcal{O}_{tmf}(f) = R$ , and  $C_f$  is a generalized elliptic curve over R classified by f.

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 $Tmf = \mathcal{O}_{tmf}(\overline{\mathcal{M}}_{ell} \rightarrow \overline{\mathcal{M}}_{ell})$ , topological modular forms.

# Introduction to Spectral Algebraic Geometry

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#### Definition

A spectral Deligne-Mumford stack is a spectral ringed  $\infty$ -topos  $X = (\mathcal{X}, \mathcal{O}_X)$  which locally likes SpétA, for an  $E_{\infty}$  ring A.

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- 3. When U be an open subset of X,  $(U, (\pi_0 \mathcal{O}_X)|_U)$  is affine.  $\pi_n(\mathcal{O}_X(U)) \to (\pi_n \mathcal{O}_X)(U)$  is an isomorphism.

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If the spectrally ringed space only satisfy the first three conditions, then we call it a nonconnective spectral scheme.

A spectral variety X over an  $E_{\infty}$ -ring R is a nonconnective spectral DM stack, such that  $\tau_{\geq 0}X \rightarrow \operatorname{Spet}\tau_{\geq 0}R$  is proper, locally almost of finite presentation, geometrically reduced and geometrically connected.

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- Abelian varieties over R : commutative monoidal objects of the  $\infty$  category Var(R).
- Strict abelian varieties over R : abelian group objects of the  $\infty$ -category Var(R).

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#### Definition

A formal spectral DM stack is a spectrally ringed  $\infty$ -topos  $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$  which admits a cover  $\{U_i\}$ , such that each  $(\mathcal{X}_{|U_i}, \mathcal{O}_{\mathcal{X}|U_i})$  is equivalent to SpfA<sub>i</sub> for some  $E_{\infty}$ -ring  $A_i$ 

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Definition

A n-dimensional formal group over a connective  $E_\infty\text{-ring}\,R$  is a functor

 $\hat{G}$  :  $\mathbf{CAlg}_R \rightarrow Mod_{\mathbb{Z}}$ 

such that the composite

$$\mathsf{CAlg}_{R} \to \mathrm{Mod}_{\mathbb{Z}} \to \mathcal{S}$$

is represented  $\operatorname{Spf}(C^{\vee})$  for some n-dimensional smooth coalgebra C.

## Theorem (Lurie, 2018)

There exists a connective  $E_{\infty}$ -ring  $R_{G_0}^{un}$  with a morphism  $\rho : R_{G_0}^{un} \to R_0$ , and a deformation G of  $G_0$  with the following properties:

•  $R_{G_0}^{un}$  is Noetherian,  $\pi_0(\rho) : \pi_0(R_{G_0}^{un}) \to R_0$  is surjective, and  $R_{G_0}^{un}$  is complete with respect to the ideal ker $(\pi_0(\rho))$ .

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- $R_{G_0}^{un}$  is Noetherian,  $\pi_0(\rho) : \pi_0(R_{G_0}^{un}) \to R_0$  is surjective, and  $R_{G_0}^{un}$  is complete with respect to the ideal ker $(\pi_0(\rho))$ .
- For other  $\rho_A: A \to R_0$  . The extension of scalars induces an equivalence of  $\infty$ -categories

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We refer to  $R_{G_0}^{un}$  as the spectral deformation ring of the p-divisible group  $G_0$ .

Let R be an  $E_{\infty}$ -ring and let X :  $CAlg_{\tau_{\geq 0}(R)}^{cn} \to S_*$  be a pointed formal hyperplane over R. A preorientation of X is a map of pointed spaces

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## Definition

A preorientation of an 1-dimensional formal group  $\hat{G}$  over a  $E_\infty\text{-ring}$  R is a map

$$e:S^2 o\Omega^\infty\hat{G}( au_{\geq 0}R)$$

of based spaces, where the based points goes to the identity of the group structure.

The dualizing line of an 1-dimensional formal group  $\hat{\boldsymbol{G}}$  is an R-module defined by

$$\omega_{\hat{G}} := \mathsf{R} \otimes_{\mathcal{O}_{\hat{G}}} \mathcal{O}_{\hat{G}}(-\eta)$$

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An orientation of a formal group is a preorientation e whose the Bott map is an equivalence.

## Theorem (Lurie, 18)

let X be a 1-dimensional pointed formal hyperplane over R. Then there exists an  $E_{\infty}$ -ring  $\mathcal{D}_X$  and  $e \in \operatorname{Or}(X_{\mathcal{D}_X})$ , such that for other  $R' \in \mathbf{CAlg}_R$ , evaluation on e induces a homotopy equivalence

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Let R be an even periodic  $E_{\infty}$ -ring, G be any formal group over R. Then there is a canonical homotopy equivalence

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The preorientation is an orientation if and only its image under the above map is a equivalence of formal groups over R.

Xuecai Ma

Doctoral Dissertation Proposal

# Applications of Spectral Algebraic Geometry

Spectral elliptic curves : spectral abelian varieties of dimension one.

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There exists a nonconnective spectral Deligne-Mumford stack  $\mathcal{M}_{\textit{ell}}^{\textit{or}}$  such that

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The elliptic spectrum has the  $E_{\infty}$  structure, since the spectral stack of oriented elliptic curve has the same underlying *étale* site with the classical stack of elliptic curve.

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Models: A class of PEL Shimura stacks (moduli stacks of abelian varieties with the extra structure of Polarization, Endomorphisms, and Level structure) which associated to a rational form of the unitary group U(1, n-1)) can give a 1-dimensional p-divisible group satisfying the conditions of this theorem.

Xuecai Ma

Doctoral Dissertation Proposal

# $\pi_* E(n) = W(k) [v_1, \cdots, v_{n-1}] [\beta^{\pm 1}]$

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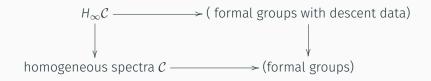
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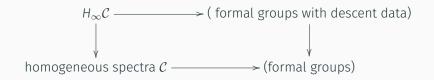
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- 5.  $E_{G_0} = L_{K_n} R_{G_0}^{or}$  is just the spectra of Morava E-theory. We refer to this as the Lubin-Tate spectrum.

# How to Lift a Complex Orientation $MU \rightarrow E$ to an $E_{\infty}$ Map



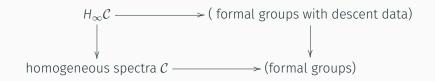


The rule which associates a level structure

 $l: A \rightarrow i^*G(R)$ 

Xuecai Ma

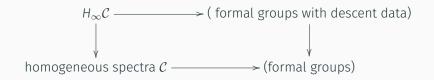
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to a map  $\psi_l^E$ : Spf $R \to S_E$  given by the ring map  $\pi_0 E \xrightarrow{D_A} \pi_0 E^{BA^*_+} \to R$ 



The rule which associates a level structure

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to a map  $\psi_l^E$ :  $\operatorname{Spf} R \to S_E$  given by the ring map  $\pi_0 E \xrightarrow{D_A} \pi_0 E^{BA^*_+} \to R$ and the isogeny

$$\psi_l^{G/E}: i^*G \to \psi_l^*G$$

is descent data for level structure on the formal group G over  $S_E$ .

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Doctoral Dissertation Proposal

 $\mathcal{L}$  is a line bundle over G. Given a subset  $I \subset \{1, \dots, k\}, \sigma_I : G_S^k \to G$  defined by  $\sigma_I(a_1, \dots, a_k) = \Sigma_{i \in I} a_i$ .

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We define a line bundle over  $G_{\rm S}^k$  by

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$$\begin{split} \Theta^{0}(\mathcal{L})_{a} &= \mathcal{L}_{a} \\ \Theta^{1}(\mathcal{L})_{a} &= \frac{\mathcal{L}_{0}}{\mathcal{L}_{a}} \\ \Theta^{2}(\mathcal{L})_{a,b} &= \frac{\mathcal{L}_{0} \otimes \mathcal{L}_{a+b}}{\mathcal{L}_{a} \otimes \mathcal{L}_{b}} \\ \Theta^{3}(\mathcal{L})_{a,b,c} &= \frac{\mathcal{L}_{0} \otimes \mathcal{L}_{a+b} \otimes \mathcal{L}_{a+c} \otimes \mathcal{L}_{b+c}}{\mathcal{L}_{a} \otimes \mathcal{L}_{b} \otimes \mathcal{L}_{c} \otimes \mathcal{L}_{a+b+c}} \end{split}$$

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- 2. s is symmetric, i.e., for each  $\sigma \in \Sigma_k$ , we have  $\xi_{\sigma} \pi_{\sigma}^* s = s$ .
- 3. The section  $s(a_1, a_2, ...) \otimes s(a_0 + a_1, a_2, ...)^{-1} \otimes s(a_0, a_1 + a_2, ...) \otimes s(a_0, a_1, ...)^{-1} \otimes$  corresponds to 1.

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If  $g: MU\langle 2k \rangle \rightarrow E$  is an orientation, then the composition

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If  $g: MU(2k) \rightarrow E$  is an orientation, then the composition

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#### Theorem

For  $0 \le k \le 3$ , the maps of ring spectra  $MU(2k) \to E$  are in one to one correspondence with  $\Theta^k$ -structures on  $\mathcal{I}(0)$  over  $G_E$ .

Let  $g: MU(2k) \to E$  be a homotopy multiplicative map,  $s = s_g$  be the section of  $\Theta^k(I_G(0))$  as before. If the map g is  $H_\infty$ , then for each level structure

$$A \stackrel{l}{\rightarrow} i^*G,$$

the section s satisfy the identity

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Using this theorem, they proved the  $\sigma$  orientation of an elliptic spectrum is an  $H_{\infty}$  map. Zhu (2020) proved that the map  $MU(0) \rightarrow E$  coming from a coordinate of  $SpfE^0(\mathbb{C}^{\infty})$  is a  $H_{\infty}$  map, since the map satisfying the condition above, which is called norm coherence.

Hopkins-Lawson obstruction theory (2018): There exists a diagram of  $E_{\infty}$ -ring spectra

$$\mathbb{S} \to MX_1 \to MX_2 \to MX_3 \to \cdots$$

such that the following hold:

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where  $F_m$  is a certain pointed space.

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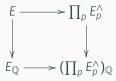
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#### Theorem (Senger, 2022)

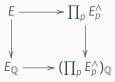
Let E denote a height  $\leq 2$  Landweber exact  $E_{\infty}$ -ring whose homotopy groups is concentrated in even degrees. Then any complex orientation  $MU \rightarrow E$  which satisfies the Ando criterion lifts uniquely up to homotopy to an  $E_{\infty}$ -ring map  $MU \rightarrow E$ . The proof of Senger's theorem was based on E-cohomology of some certain spaces.

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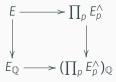
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It suffices to lift the induced complex orientation of  $E_p^{\wedge}$ . Assume that E is p-complete. So we only need to prove

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and a equivalence

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$$E^1(\Sigma^{\infty}F_{p^2})\simeq 0$$

Lemma (Senger, 2022)  $E^{2n}(F_p) \cong E^{2n+1}(F_{p^2}) \cong 0.$  Lemma (Senger, 2022)  $E^{2n}(F_p) \cong E^{2n+1}(F_{p^2}) \cong 0.$ 

Let  $L_m$  denote the nerve of the poset of proper direct sum decomposition of  $\mathbb{C}^m$ , and  $(L_m)^{\diamond}$  is its unreduced suspension.

$$F_m \simeq ((L_m)^\diamond \wedge S^{2m})_{hU(m)}.$$

• What is the conceptional description of the complex orientation in the context of spectral algebraic geometry? What is the relation between the spectral Quillen formal group and level structures?

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- The descent data of  $H_{\infty}$ -spectrum only consider the level one structures, what about the infinity level structures?
- Norm coherence condition in the context of spectral algebraic geometry.

# Thanks for Your Listening !

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# Questions and Answers !

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