Methods of Spectral Algebraic Geometry in Chromatic Homotopy Theory

Doctoral Dissertation Proposal

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$$\begin{array}{ccc} E \otimes E & \longrightarrow & E \\ {\scriptstyle f\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ F \otimes F & \longrightarrow & F \end{array}$$

Formal Groups

Let R be a complete local ring with residue filed characteristic $p > 0$, C_R denote the category of local Noetherian R-algebras. We define

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$$\mathcal{O}_G \to \mathcal{O}_{G\times G} \cong \mathcal{O}_G \otimes \mathcal{O}_G$$

$\mathcal{O}_G$ is just $R[\![X]\!]$ and $\mathcal{O}_G \otimes \mathcal{O}_G$ is $R[\![X]\!] \otimes_R R[\![Y]\!] = R[\![X, Y]\!]$.

$$\begin{array}{rccc} \phi : & R[\![X]\!] & \to & R[\![X, Y]\!] \\ & X & \to & f(X, Y) \end{array}$$

Formal Group Laws

Definition

Formal group law : $F \in R[\![x_1, x_2]\!]$

- $F(x, 0) = F(0, x) = x$ (Identity)
- $F(x_1, x_2) = F(x_2, x_1)$ (Commutativity)
- $F(F(x_1, x_2), x_3) = F(x_1, F(x_2, x_3))$ (Associativity)

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There exists a ring L and $F_{univ}(x, y) \in L[\![x, y]\!]$

$$\{\text{Formal Group Law over R}\} \longleftrightarrow \{L \to R\}$$

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Lazard's Theorem

$L \cong \mathbb{Z}[t_1, t_2, \cdots]$

Heights of Formal Groups

Let $f(x, y) \in R[\![x, y]\!]$

1. If $n = 0$,we set $[n](t) = 0$.
2. If $n > 0$, we set $[n](t) = f([n-1](t), t)$.

P-series $p[t]$ is either 0 or equals $\lambda t^{p^n} + O(t^{p^n+1})$ for some $n > 0$.

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Examples

1. Formal multiplicative group $f(x, y) = x + y + xy$, $[n](t) = (1+t)^n - 1$. If $p = 0$ in R, then $[p](t) = (1+t)^p - 1 = t^p$, so f has height 1.

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2. Formal additive group $f(x, y) = x + y$, if $p = 0$ in R. Then $[p](t) = 0$, so f has infinite height.

Complex Oriented Cohomology Theories

Complex Orientation

Let E be cohomology theory. Then a complex orientation of E is a choice $x \in E^2(\mathbb{C}P^\infty)$ whcih restricts to 1 under the composite

$$E^2(\mathbb{C}P^\infty) \to E^2(\mathbb{C}P^1) = E^2(S^2) \cong E^0(*)$$

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$$E^*(\mathbb{CP}^\infty) \cong E^*(*)[\![t]\!] = (\pi_* E)[[t]]$$

$$(\pi_* E)[[t]] \cong E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong (\pi_* E)[[x, y]]$$

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Theorem(Quillen, 1969)

MU is the universal complex oriented cohomology theory,
$L \cong \pi_* \mathrm{MU}$.

For E complex oriented, $MU \to E$, induce $L = \pi_* MU \to \pi_* E$.

If we already have a ring map $L \to R$, can we construct a complex oriented cohomology theory E such that $R = \pi_* E$?

The Landweber Exact Functor Theorem

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Landweber's Exact Functor Theorem, 1976

Let M be a module over the Lazard ring L. Then M is flat over $\mathcal{M}_{FG}$ if and only if for every prime number p, the elements $v_0 = p, v_1, v_2, \cdots \in L$ form a regular sequence for M.

Lubin-Tate Theory

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Lubin-Tate's Theorem, 1966

There is a universal formal group G over $R_{LT} = W(k)[[v_1, \cdots, v_n - 1]]$ in the following sense: for every infinitesimal thickening A of k, there is a bijection

$$\mathrm{Hom}_{/k}(R_{LT}, A) \to \mathrm{Def}(A).$$

Morava E-theories and Morava K-theories

Using Landweber exact functor theorem, there is a even periodic spectrum $E(n)$

$$\pi_* E(n) = W(k)[\![v_1, \cdots, v_{n-1}]\!][\beta^{\pm 1}]$$

Theorem (Goerss-Hopkins-Miller, 2004)

The spectrum E(n) admits a unique E_∞-ring structure.

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The spectrum E(n) admits a unique E_∞-ring structure.

$M(k)$ denote the cofiber of the map $\sum^{2k} MU_{(p)} \to MU_{(p)}$ given by the multiplication by t_k.

Let $K(n)$ denote the smash product

$$MU_{(p)}[v_n^{-1}] \otimes_{MU_{(p)}} \bigotimes_{k \neq p^n - 1} M(k).$$

This spectrum $K(n)$ is called **Morava K-theory**. The homotopy groups of $K(n)$ is

$$\pi_* K(n) \cong (\pi_* MU_{(p)})[v_n^{-1}]/(t_0, t_1, \cdots t_{p^n-2}, t_{p^n}, \cdots) \cong \mathbb{F}_p[v_n^{\pm 1}]$$

Bousfield localization

C_E the collection of E-acyclic spectra. A spectrum is E-local if every map for every $Y \in C_E$, the map $Y \to X$ is nullhomotopic.

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- $L_{E(n)}$, behaves like restriction to the open substack $\mathcal{M}_{FG}^{\leq n} \subset \mathcal{M}_{FG} \times \mathrm{Spec}\mathbb{Z}_{(p)}$.
- $L_{K(n)}$, behaves like completion along the locally closed substack $\mathcal{M}_{FG}^{n} \subset \mathcal{M}_{FG} \times \mathrm{Spec}\mathbb{Z}_{(p)}$.

Elliptic Cohomology

An elliptic cohomology consists of

1. An even periodic spectrum E .
2. An elliptic curve C over $\pi_0 E$.
3. $\phi : G_E \cong \hat{C}$

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Theorem(Goerss-Hopkins-Miller-Lurie)

There is a sheaf $\mathcal{O}_{tmf}$ of E_∞-ring spectra over the stack $\overline{\mathcal{M}}_{ell}$ for the *étale* topology. For any *étale* morphism $f : \mathrm{Spec}(R) \to \overline{\mathcal{M}}_{ell}$, there is a natural structure of elliptic spectrum $(\mathcal{O}_{tmf}(f), C_f, \phi)$, satisfying $\pi_0\mathcal{O}_{tmf}(f) = R$, and C_f is a generalized elliptic curve over R classified by f.

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$Tmf = \mathcal{O}_{tmf}(\overline{\mathcal{M}}_{ell} \to \overline{\mathcal{M}}_{ell})$, topological modular forms.

Introduction to Spectral Algebraic Geometry

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1. $\pi_0 M$ is *étale* over $\pi_0 A$..
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Spectral Stacks

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Definition

A spectral Deligne-Mumford stack is a spectral ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_X)$ which locally likes $\mathrm{Sp\acute{e}t} A$, for an E_∞ ring A.

Spectral Schemes

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3. When U be an open subset of X, $(U, (\pi_0\mathcal{O}_X)|_U)$ is affine. $\pi_n(\mathcal{O}_X(U)) \to (\pi_n\mathcal{O}_X)(U)$ is an isomorphism.

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If the spectrally ringed space only satisfy the first three conditions, then we call it a nonconnective spectral scheme.

Spectral Varieties

Definition

A spectral variety X over an E_∞-ring R is a nonconnective spectral DM stack, such that $\tau_{\geq 0} X \to \mathrm{Spet} \tau_{\geq 0} R$ is proper, locally almost of finite presentation, geometrically reduced and geometrically connected.

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- Strict abelian varieties over R : abelian group objects of the ∞-category $\mathrm{Var}(R)$.

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A formal spectral DM stack is a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ which admits a cover $\{U_i\}$, such that each $(\mathcal{X}_{|U_i}, \mathcal{O}_{\mathcal{X}|U_i})$ is equivalent to $\mathrm{Spf} A_i$ for some E_∞-ring A_i

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A n-dimensional formal group over a connective E_∞-ring R is a functor

$$\hat{G} : \mathsf{CAlg}_R \to \ Mod_{\mathbb{Z}}$$

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Deformations

G_0 be a p-divisible group over R_0. A deformation of G_0 along $\rho_A : A \to R_0$ is a pair (G, α), where G is a spectral p-divisible group over A and $\alpha : G_0 \simeq \rho_A^* G$.

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Theorem (Lurie, 2018)

There exists a connective E_∞-ring $R_{G_0}^{un}$ with a morphism $\rho : R_{G_0}^{un} \to R_0$, and a deformation G of G_0 with the following properties:

- $R_{G_0}^{un}$ is Noetherian, $\pi_0(\rho) : \pi_0(R_{G_0}^{un}) \to R_0$ is surjective, and $R_{G_0}^{un}$ is complete with respect to the ideal $\ker(\pi_0(\rho))$.

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There exists a connective E_∞-ring $R^{un}_{G_0}$ with a morphism $\rho : R^{un}_{G_0} \to R_0$, and a deformation G of G_0 with the following properties:

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- For other $\rho_A : A \to R_0$. The extension of scalars induces an equivalence of ∞-categories

$$\mathrm{Map}_{\mathsf{CAlg}_{/R_0}}(R^{un}_{G_0}, A) \to \mathrm{Def}_{G_0}(A, \rho_A).$$

Deformations

G_0 be a p-divisible group over R_0. A deformation of G_0 along $\rho_A : A \to R_0$ is a pair (G, α), where G is a spectral p-divisible group over A and $\alpha : G_0 \simeq \rho_A^* G$.

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We refer to $R^{un}_{G_0}$ as the *spectral deformation ring* of the p-divisible group G_0.

Definition

Let R be an E_∞-ring and let $X : \mathrm{CAlg}^{cn}_{\tau_{\geq 0}(R)} \to \mathcal{S}_*$ be a pointed formal hyperplane over R. A preorientation of X is a map of pointed spaces

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Orientations

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Definition

A preorientation of an 1-dimensional formal group $\hat{G}$ over a E_∞-ring R is a map

$$e : S^2 \to \Omega^\infty \hat{G}(\tau_{\geq 0} R)$$

of based spaces, where the based points goes to the identity of the group structure.

The dualizing line of an 1-dimensional formal group $\hat{G}$ is an R-module defined by

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Definition

An orientation of a formal group is a preorientation e whose the Bott map is an equivalence.

Theorem (Lurie, 18)

let X be a 1-dimensional pointed formal hyperplane over R. Then there exists an E_∞-ring $\mathcal{D}_X$ and $e \in \mathrm{Or}(X_{\mathcal{D}_X})$, such that for other $R' \in \mathsf{CAlg}_R$, evaluation on e induces a homotopy equivalence

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Lemma

Let R be an even periodic E_∞-ring, G be any formal group over R. Then there is a canonical homotopy equivalence

$$\mathrm{Pre}G \simeq \mathrm{Map}_{FG(R)}(G_R^Q, G)$$

Where G_R^Q is the spectral Quillen formal group, whose 0-th homotopy is the classical Quillen formal group.

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Where G_R^Q is the spectral Quillen formal group, whose 0-th homotopy is the classical Quillen formal group.

The preorientation is an orientation if and only its image under the above map is a equivalence of formal groups over R.

Applications of Spectral Algebraic Geometry

Elliptic Cohomology Theory

Spectral elliptic curves : spectral abelian varieties of dimension one.

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There exists a nonconnective spectral Deligne-Mumford stack $\mathcal{M}_{ell}^{or}$ such that

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The elliptic spectrum has the E_∞ structure, since the spectral stack of oriented elliptic curve has the same underlying *étale* site with the classical stack of elliptic curve.

Topological Automorphic Forms

Theorem (Lurie, 2010-2017)

Let M_{pd}^{n} denote the moduli stack of one dimensional height n p-divisible group, then there is a sheaf of E_∞-ring space $\mathcal{O}^{top}$ on the *étale* site,

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Models: A class of PEL Shimura stacks (moduli stacks of abelian varieties with the extra structure of Polarization, Endomorphisms, and Level structure) which associated to a rational form of the unitary group U(1, n–1)) can give a 1-dimensional p-divisible group satisfying the conditions of this theorem.

$$\pi_* E(n) = W(k)[\![v_1, \cdots, v_{n-1}]\!][\beta^{\pm 1}]$$

1. $\hat{G}_0$ is a formal group over k, viewed as a identity component of a connected p-divisible group G_0.

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E_∞ structures of Morava E-theories

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4. Let $R^{or}_{G_0}$ be an orientation classifier for G^0_{un}.
5. $E_{G_0} = L_{K_n} R^{or}_{G_0}$ is just the spectra of Morava E-theory. We refer to this as the Lubin-Tate spectrum.

How to Lift a Complex Orientation $MU \to E$ to an E_∞ Map

Obstructions $\to H_\infty$-maps

$$
\begin{array}{ccc}
H_\infty\mathcal{C} & \longrightarrow & (\text{ formal groups with descent data}) \\
\downarrow & & \downarrow \\
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to a map $\psi_l^E : \mathrm{Spf} R \to S_E$ given by the ring map $\pi_0 E \xrightarrow{D_A} \pi_0 E^{BA_+^*} \to R$ and the isogeny

$$\psi_l^{G/E} : i^*G \to \psi_l^* G$$

is descent data for level structure on the formal group G over S_E.

$\mathcal{L}$ is a line bundle over G. Given a subset $I \subset \{1, \cdots, k\}$, $\sigma_I : G_S^k \to G$ defined by $\sigma_I(a_1, \ldots, a_k) = \Sigma_{i \in I} a_i$.

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We define a line bundle over G_S^k by

$$\Theta^k(\mathcal{L}) = \bigotimes_{I \subset \{1,\ldots,k\}} (\mathcal{L}_I)^{(-1)^{|I|}}$$

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$$\begin{aligned}
\Theta^0(\mathcal{L})_a &= \mathcal{L}_a \\
\Theta^1(\mathcal{L})_a &= \frac{\mathcal{L}_0}{\mathcal{L}_a} \\
\Theta^2(\mathcal{L})_{a,b} &= \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b}}{\mathcal{L}_a \otimes \mathcal{L}_b} \\
\Theta^3(\mathcal{L})_{a,b,c} &= \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b} \otimes \mathcal{L}_{a+c} \otimes \mathcal{L}_{b+c}}{\mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c \otimes \mathcal{L}_{a+b+c}}
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3. The section $s(a_1, a_2, \dots) \otimes s(a_0 + a_1, a_2, \dots)^{-1} \otimes s(a_0, a_1 + a_2, \dots) \otimes s(a_0, a_1, \dots)^{-1} \otimes$ corresponds to 1.

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If $g : MU\langle 2k\rangle \to E$ is an orientation, then the composition

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Theorem

For $0 \leq k \leq 3$, the maps of ring spectra $MU\langle 2k \rangle \to E$ are in one to one correspondence with Θ^k-structures on $\mathcal{I}(0)$ over G_E.

Theorem (Ando-Hopkins-Strickland, 2004)

Let $g : MU\langle 2k\rangle \to E$ be a homotopy multiplicative map, $s = s_g$ be the section of $\Theta^k(I_G(0))$ as before. If the map g is H_∞, then for each level structure

$$A \xrightarrow{l} i^*G,$$

the section s satisfy the identity

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Using this theorem, they proved the σ orientation of an elliptic spectrum is an H_∞ map. Zhu (2020) proved that the map $MU\langle 0\rangle \to E$ coming from a coordinate of $\mathrm{Spf} E^0(\mathbb{C}^\infty)$ is a H_∞ map, since the map satisfying the condition above, which is called norm coherence.

Obstructions to E_∞-maps

Hopkins-Lawson obstruction theory (2018): There exists a diagram of E_∞-ring spectra

$$\mathbb{S} \to MX_1 \to MX_2 \to MX_3 \to \cdots$$

such that the following hold:

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2. $\mathrm{Map}_{E_\infty}(MX_1, E) \simeq Or(E)$ for each E_∞-ring E.
3. Given $m > 0$ and an E_∞-ring E, there is a pull back square

$$\begin{array}{ccc} \mathrm{Map}_{E_\infty}(MX_m, E) & \longrightarrow & \mathrm{Map}_{E_\infty}(MX_{m-1}, E) \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \mathrm{Map}_*(F_m, \mathrm{Pic}(E)) \end{array}$$

where F_m is a certain pointed space.

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5. Let E denote an E_∞ such that $\pi_* E$ is p-local and p-torsion free. Then an E_∞-map $MX_1 \to E$ extends to an E_∞ map $MX_P \to E$ if and only if the corresponding complex orientation of E satisfies the Ando criterion.

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Theorem (Senger, 2022)

Let E denote a height ≤ 2 Landweber exact E_∞-ring whose homotopy groups is concentrated in even degrees. Then any complex orientation $MU \to E$ which satisfies the Ando criterion lifts uniquely up to homotopy to an E_∞-ring map $MU \to E$.

The proof of Senger's theorem was based on E-cohomology of some certain spaces.

We have the following pullback square.

$$\begin{array}{ccc} E & \longrightarrow & \prod_p E_p^\wedge \\ \downarrow & & \downarrow \\ E_{\mathbb{Q}} & \longrightarrow & (\prod_p E_p^\wedge)_{\mathbb{Q}} \end{array}$$

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\pi_0 \mathrm{Map} E_\infty(MU, R) & \longrightarrow & \pi_0 \mathrm{Map}_{E_\infty}(MU, \prod_p E_P^{\wedge}) \\
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\pi_0 \mathrm{Or}(E_{\mathbb{Q}}) & \longrightarrow & \pi_0 \mathrm{Or}((\prod_p E_p^{\wedge})_{\mathbb{Q}})
\end{array}
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\begin{array}{ccc}
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It suffices to lift the induced complex orientation of $E_p^\wedge$.

Assume that E is p-complete. So we only need to prove

$$\pi_0 \mathrm{Map}_{E_\infty}(MX_{p^2}, E) \to \pi_0 \mathrm{Map}_{E_\infty}(MX_p, E)$$

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$$\mathrm{Map}_{E_\infty}(MX_{p^2}, E) \to \mathrm{Map}_{E_\infty}(MX_p, E) \to \mathrm{Map}_*(F_{p^2}, Pic(E))$$

and a equivalence

$$\mathrm{Map}_{E_\infty}(F_m, Pic(E)) \simeq \mathrm{Hom}(\Sigma^\infty F_m, pic(E)) \simeq \mathrm{Hom}(\Sigma^\infty F_m, \Sigma E).$$

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It suffices to show that

$$E^1(\Sigma^\infty F_{p^2}) \simeq 0$$

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Let L_m denote the nerve of the poset of proper direct sum decomposition of $\mathbb{C}^m$, and $(L_m)^\diamond$ is its unreduced suspension.

$$F_m \simeq ((L_m)^\diamond \wedge S^{2m})_{hU(m)}.$$

Plan

Our question is how to lift a complex orientation $MU \to E$ to an E_∞-map ? Especially when E is a Morava E-theory.

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Thanks for Your Listening !

Questions and Answers !