$\mathbb{C}((X))$-REPRESENTATIONS OF GL_n OVER 2-DIMENSIONAL LOCAL FIELDS

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Abstract. Using the $\mathbb{R}((X))$-measure, we define and study spaces of certain $\mathbb{C}((X))$-valued functions on $\mathrm{GL}_n(F)$ for F a two-dimensional local field. In particular, we define a convolution product on one of the function space, which leads us to define the Hecke algebra of $\mathrm{GL}_n(F)$. We define and study the measurable $\mathbb{C}((X))$-representation of $\mathrm{GL}_n(F)$, and prove that function space is a candidate of measurable representation.

Contents

1. Introduction

The concept of an n-dimensional field was first introduced by Parshin in 1970s, which aims to generalizing the classical adelic formalism to n-dimensional schemes. An n-dimensional field is a complete discrete field with residue field an $(n-1)$-dimensional local field. A 0-dimensional field is just a finite field.

It is an important topic in number theory to study representations of algebraic groups over 0- and 1-dimensional local fields. Since finite fields and local fields have good properties, for example, the local fields are topological fields, people know many things about their representations. But for algebraic groups over two-dimensional local fields, people know little about their representations. In [Kap01], Kapranov study the central extension Γ of a reductive group G over a two-dimensional local field $F = K((t))$, where K is an 1-dimensional local field. They choose an appropriate subgroup $\Delta_1 \subset \Gamma$, such that the fibres of the Hecke correspondences are locally compact spaces which can define invariant measures, so they can define the Hecke operators by integrating these measures. They proved that such Hecke algebra $H(\Gamma, \Delta_1)$ is isomorphic to the double affine Hecke algebra associated to G. After that, Gaitsgory and Kazhdan give a categorical framework of Kapranov's idea. They study representations in pro-vector spaces, see [GK04, GK05, GK06] for more details and [BK06] for more examples.

Date: September 18, 2025.

On the other hand, I. Fesenko constructed a $\mathbb{R}((X))$-measure on two-dimensional local fields [Fes03], such that we can do harmonic analysis on two-dimensional local fields. H. Kim and K. H. Lee use the Cartan decompositions to define the generators and relations of spherical Hecke algebra of SL_2 [KL04]. In particular, they proved the Satake isomorphism by using the $\mathbb{R}((X))$-measures. And one can see [Lee10] for the construction of Iwahori-Hecke Algebra for SL_2 over a 2-dimensional local field.

There is another strategy to study Hecke algebra of reductive groups over two-dimensional local fields. In [BK11], Braverman and Kazhdan consider the subgroup G^+_{aff} of G_{aff}, where G_{aff} is the semidirect product of G_m and the central extension $\widetilde{G}$ of $G((t))$. They proved that any double cosets of $G_{\text{aff}}(\mathcal{O})$ inside G^+_{aff} is well defined and give rise to an algebra structure on a suitable space of $G_{\text{aff}}(\mathcal{O})$-biinvariant functions on $G^+_{\text{aff}}(K)$. They call it the spherical Hecke algebra of G_{aff}. And in [BKP16], the authors use $W_X = W \ltimes X$, the semidirect product of the Weyl group with the Tits cone, to identify the double cosets $I \setminus G^+_{\text{aff}}/I$. By studying the combinatorics of the Tits cone X, they define the Iwahori-Hecke algebra of G_{aff}.

The present paper gives a new construction of Hecke algebra of $\mathrm{GL}_n(F)$ for F a two-dimensional local field by using Fesenko's $\mathbb{R}((x))$-measure and Morrow's work [Mor08] on integration on $\mathrm{GL}_n(F)$. Specifically, we define an appropriate space of functions $f : \mathrm{GL}_n(F) \to \mathbb{C}((X))$ and prove that there is a well-defined convolution product in this space. We define and study a kind of representation called measurable representations. The typical example is the function space that we defined. We also introduce the automorphic forms on 2-dimensional adelic spaces defined by I. Fesenko.

We now give the structure of this paper. In section 2, we first review the basic definition and properties of higher-dimensional local fields. After that, we review the topology and measures on two-dimensional local fields. We then introduce the integration on $\mathrm{GL}_n(F)$ for F a two-dimensional local field. We define the spaces $\mathcal{L}^H(\mathrm{GL}_n(F))$ consisting of functions $f : \mathrm{GL}_n(F) \to \mathbb{C}((X))$ satisfing certain conditions. We define the convolution product on $\mathcal{L}^H(\mathrm{GL}_n(F))$, make $\mathcal{L}^H(GL_n(F))$ be an associative algebra, called the Hecke algebra of $\mathrm{GL}_n(F)$.

In section 3, we define the measurable representations of $\mathrm{GL}_n(F)$, and prove that the function space $\mathcal{L}^H\mathrm{GL}_n(F)$ is a measurable representation. This makes us establish a relation between the Hecke modules with measurable representations.

In section 4, we introduce the adelic spaces over 2-dimensional arithmetic objects. We then introduce the automorphic forms over two-dimensional adelic spaces suggested by Fesenko.

Acknowledgements. The author thanks Ivan Fesenko for his support of this research and discussions about related topics of this work. The author also thanks Kye-Huan. Lee for helpful discussions. This work was supported by ***** grant ********.

2. Integration on $\mathrm{GL}_n(F)$

2.1. **Definition of two-dimensional local fields.**

Definition 2.1. A 0-dimensional local field is a finite field. For $n \geq 1$, an n-dimensional local field is a complete discrete valuation field whose residue field is an $(n-1)$-dimensional local field.

Example 2.2. One-dimensional local fields

(1) $\mathbb{R}$, $\mathbb{C}$;
(2) $\mathbb{F}_q((t))$;
(3) Finite extension of $\mathbb{Q}_p$.

Example 2.3. Two-dimensional local fields

(1) $\mathbb{F}_q((t_1))((t_2))$;
(2) $E((t))$ over a local nonarchimedean field E;
(3) $E((t))$ over a local archimedean field E;
(4) Finite extensions of $\mathbb{Q}_p\{\{t\}\}$.

Theorem 2.4. *Let F be an n-dimensional local field.*

(1) *If* $\mathrm{char}F \neq 0$ *then*
$$F \cong F^{(n)}((t_1))\cdots((t_n)),$$
where $F^{(n)}$ is a finite field.
(2) *If* $\mathrm{char}F^{(n-1)} = 0$ *then*
$$F \cong F^{(n-1)}((t_1))\cdots((t_{n-1})),$$
where $F^{(n-1)}$ is a one-dimensional local field of characteristic 0.
(3) *In the remaining case, Let $2 \leq r \leq n$ be the unique integer such that* $\mathrm{char}F^{(n-r)} = 0 \neq \mathrm{char}F^{(n+1-r)}$. *Then F is isomorphic to a finite extension of*
$$\mathbb{Q}\{\{t_1\}\}\cdots\{\{t_{r-1}\}\}((t_{r+1}))\cdots((t_n))$$
where $\mathbb{Q}_q$ is the unramified extension of $\mathbb{Q}_p$ with residue field of $F^{(n)}$.

If $n = 0$, we define $\mathcal{O}_F^0 = F$. If $n \geq 0$, we define $\mathcal{O}_F^n := \{x \in \mathcal{O}_F : \overline{x} \in \mathcal{O}_{\overline{F}}^{(n-1)}\}$, where $\mathcal{O}_{\overline{F}}^{(n-1)}$ is the rank $n-1$ ring of integers of $\overline{F}$, a field of discrete valuation of dimension $\geq n-1$.

$$F \supset \mathcal{O}_F = \mathcal{O}_F^{(1)} \supset \mathcal{O}_F^{(2)} \supset \cdots \supset \mathcal{O}_F^{(n)}.$$

Suppose that F is a complete valuation field, and $t \in F$ is uniformizer, then we have
$$F = \mathcal{O}_F[t^{-1}], \quad F^\times \cong \mathcal{O}_F^\times \times t^{\mathbb{Z}},$$
where $t^{\mathbb{Z}}$ denote the infinite cyclic group of $F^\times$ generated by t.

Definition 2.5. A sequence of n-local parameters $t_1, \cdots, t_n \in F$ is a sequence of elements satisfying:

(1) t_n is a uniformizer of F.
(2) The reduction $t_1, \cdots, t_{n-1}$ of $t_1, \cdots, t_{n-1}$ form a sequence of local parameter for the field $\bar{F}$ of dimension $n-1$.

Proposition 2.6. *Suppose that F is a n-dimensional local field , then we have*
$$F = O_F^{(n)}[t_1^{-1}, \cdots, t_n^{-1}], \text{ and } F^\times \cong (O_F^n)^\times \times t_1^{\mathbb{Z}} \times \cdots \times t_n^{\mathbb{Z}}.$$

2.2. **Topology of two-dimensional local fields.** For a p-adic local field K, the topology of K is the p-adic topology whose basic open neighbourhoods of 0 are $\{x||x| \leq 1/p^n\}_{n\geq 0}$.

Lemma 2.7. *Let X be a Hausdorff topological space, following conditions are equivalent:*

1. *X is locally compact and totally disconnected;*
2. *X has a basis consisting of open compact sets.*
3. *Each point of X has an open neighbourhood that is a profinite space.*

Equal characteristic case Suppose that K is a field equipped with a topology satisfies the following conditions:

1. The topology is Hausdorff;
2. The addition, and multiplication by a fixed element $\alpha \in K$, are all continuous maps

Following [Par84], we can define a topology on 2-dimensional local fields.

Definition 2.8. Let $F = K((t))$ be the field of formal Laurent series over K. We define a topology on F by setting the basic neighbourhoods of 0 are those of the form

$$\sum_i U_i t^i := \left\{ \sum_i a_i t^i \in F | a_i \in U_i \text{ for all } i \right\},$$

where $(U_i)_{i=-\infty}^{\infty}$ are open neighbourhoods of 0 such that $U_i = K$ for all $i \gg 0$. The basic open neighbourhoods of any other point $f \in F$ are $f + U$, where U_i are basic neighbourhoods of 0.

Remark 2.9.
1. If the topology on K is the discrete topology. Then the topology of $K((t))$ just described will be the usual discrete valuation topology.
2. If $F \cong K((t))$, then the topology of F may depend on the choice of isomorphism.
3. If K is a one-dimensional local field, such as $\mathbb{Q}_p$. They are all topological fields. But $K((t))$ is not a topological field, since the multiplication and inverse are not continuous.

If we have an algebraic group G, the above proposition makes $G(F)$ become a topological group. But if F is a two-dimensional local field, F is not a topological field. It's difficult to discuss the topology of $G(F)$, which makes $G(F)$ be a topological group.

2.3. **Measures and integrations of two-dimensional local fields.** In the following contents, F is a two-dimensional local field with first residue field E and second residue field F_q, rank one integers $\mathcal{O}$, rank two integers O, local parameters are t_1, t_2.

Let $\mathcal{A}$ be the minimal ring generated by the distinguished sets $\{\alpha + t_2^i t_1^j O\}$. Elements of $\mathcal{A}$ can be written as a finite disjoint union sets A_n, where each A_n is a different between a distinguished set and a finite union of distinguished sets.

Proposition 2.10. ([Fes03]) *There is a unique measure μ on $\mathcal{F}$ with values in $\mathbb{R}((X))$ which is a translation invariant and finitely addictive such that*

$$\mu(t_2^i t_1^j O) = q^{-j} X^i.$$

Example 2.11. We have $\mu(O)=1$ and $\mu(t_2^i p^{-1}(S)) = X^i\mu_E(S)$, where μ_E is the normalized Haar measure on E such that $\mu_E(\mathcal{O}(E))=1$ and S is a compact open subsets of E.

We call a sum

$$\sum_n\sum_i a_{i,n}X^i$$

is absolutely convergent if

(1) there is i_0 such that $a_{i,n}=0$ for all $i\le i_0$ and all n;
(2) for every i the series $\sum_n a_{i,n}$ absolutely convergent in $\mathbb{C}$.

Remark 2.12. The measure on F is additive in the following sense: suppose that $\{A_n\}$ are countably many disjoint sets of $\mathcal{A}$ such that $UA_n\in\mathcal{A}$ and $\sum\mu(A_n)$ absolutely converges in $\mathbb{C}((X))$.

Having a measure of F, we can define the integrations. We consider functions $f: F\to\mathbb{C}((X))$.

(1) For any $A\in\mathcal{A}$, we can consider the char_A, we define
$$\int_F \mathrm{char}_A = \mu(A)$$
(2) For functions f can be written as $\sum c_n\mathrm{char}_{A_n} + \sum a_i \mathrm{char}_{\{p_i\}}$, where $\{A_n\}$ are countably disjoint measurable sets in $\mathcal{A}$, $c_n\in\mathbb{C}((X))$ such that $\sum c_n\mu(A_n)$ is absolutely converges in $\mathbb{C}((X))$, and $\{p_1,\cdots,p_m\}\}$ is a collection of finitely many points. We define
$$\int_F(\sum c_n\mathrm{char}_{A_n} + \sum_{i=1}^k a_i\mathrm{char}_{\{p_i\}})d\mu = \sum c_n\mu(A_n).$$

Since $\mathcal{A}$ contains $a+t_2^j p^{-1}(S)$, for S a compact open subsets in E. We have

$$\int_F \mathrm{char}_{a+t_2^j p^{-1}(S)}d\mu = \mu_E(S)X^j.$$

If we suppose the coefficients $c_k\in\mathbb{C}$. Then

$$\int_F\sum_k c_k\mathrm{char}_{a+t_2^i t_1^{j_k}O}d\mu = \left(\sum c_k q^{-j_k}\right)X^i = \left(\int_E\sum c_k\mathrm{char}^E_{t_1^{j_k}O_E}d\mu_E\right)X^i = \left(\int_E f d\mu_E\right)X^i,$$

for some $f\in\mathcal{L}(E)$. The integration defined above can also be defined by the following setting, see [Mor10] for more details.

(1) First, we let $\mathcal{L}(E)$ denote the space of locally constant integrable functions g on E.
(2) For $g\in\mathcal{L}(E)$, $a\in F, i\in\mathbb{Z}$, we define a function on F by
$$g^{a,i}(x) = \begin{cases} g\circ p((x-a)t_2^{-i}) & x\in a+t_2^iE[[t_2]] \\ 0 & \text{otherwise}\end{cases}$$
(3) A simple function on F is a $\mathbb{C}(((X))$-valued function of the form
$$x\mapsto g^{a,i}(x)X^j$$
for some $g\in\mathcal{L}(E)$, $a\in F$, $i,j\in\mathbb{Z}$.
(4) Let $\mathcal{L}(F)$ denote the space of all $\mathbb{C}((X))$-valued functions spanned by simple functions.

(5) For simple functions, define

$$\int_F g^{a,i}(x)dx = (\int_E g(u)du)X^i,$$

and extend it to all functions in $\mathcal{L}(F)$.

2.4. **Integrations on GL_n of a two-dimensional local field.** We first review the approach of [Mor08]. We consider the integration on F^n The important thing of repeated integral is the Fubini property, says that for a function

$$f : F^n \to \mathbb{C}((X)),$$

and for each permutation σ of $\{1, \cdots, n\}$, the expression

$$\int^F \cdots \int^F f(x_1, \cdots, x_n)dx_{\sigma(1)} \cdot dx_{\sigma(n)}$$

is well defined and its value doesn't depend on σ.

(1) Let $\mathcal{L}(E^n)$ denote the space of functions $f : E^n \to \mathbb{C}$ which are Fubini.
(2) For every $g \in \mathcal{L}(E^n)$, and every $(a_1+\cdots+a_n) \in F^n$, $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}^n$. The product of translated fraction ideals is given by

$$a + t_2^{\gamma}\mathcal{O}_F^n = \prod a_i + t_2^{\gamma_i}\mathcal{O}_F \quad \in F^n.$$

We define

$$g^{a,\gamma}(x) = \begin{cases} g(\overline{(x-a)t_2^{-\gamma}}) & x \in a + t_2^{\gamma}\mathcal{O}_F^n \\ 0 & \text{otherwise.} \end{cases}$$

(3) We define

$$\int_{F^n} g^{a,\gamma}(x)dx = \left(\int_{E^n} g d\mu_{E^n}\right) X^{\sum_{i=1}^n \gamma_i}.$$

(4) A complex function $f : E^N \to \mathbb{C}$ is called GL-Fubini if and only $f \circ \tau$ is Fubini for all $\tau \in \mathrm{GL}_n(F)$. Let $\mathcal{L}(F^n, \mathrm{GL}_n)$ denote the $\mathbb{C}((X))$ space-valued functions spanned by

$$g^{a,\gamma} \circ \tau, \quad \tau \in \mathrm{GL}_n(F), a \in F^n, \gamma \in \mathbb{Z}^n.$$

Theorem 2.13. *([Mor08, Theorem 3.4]) Every function in $\mathcal{L}(F^n, \mathrm{GL}_n)$ is Fubini on F^n. If $f \in \mathcal{L}(F^n, GL_n), a \in F^n, \tau \in \mathrm{GL}_n(F)$, then the functions $x \mapsto f(x+a)$ and $x \mapsto f(\tau x)$ also belongs to $\mathcal{L}(F^n, \mathrm{GL}_n)$, and the repeated integrals are given by*

$$\int_{F^n} f(x+a)dx = \int_{F^n} f(x)dx,$$

$$\int_{F^n} f(\tau x)dx = |\det \tau|^{-1} \int_{F^n} f(x)dx.$$

Now, we consider the integration on $\mathrm{GL}_n(F)$. Let $T : F^{n^2} \to M_N(F)$ be the isomorphism from the vector space F^{n^2} to the vector space of $n \times n$ matrices. Let $\mathcal{L}(M_n(F))$ be the space of $\mathbb{C}((X))$-valued functions f such that $f \circ T$ belongs to $\mathcal{L}(F^n, \mathrm{GL}_n)$, and we define

$$\int_{M_n(F)} f(x)dx = \int_{F^{n^2}} f \circ T(x)dx.$$

Let $\mathcal{L}(G_n(F))$ denote the space of $\mathbb{C}((X))$-valued functions ϕ on $\mathrm{GL}_n(F)$ such that $\tau \mapsto \phi(\tau)|\det \tau|^{-n}$ extends to a function of $\mathcal{L}(M_n(F))$. We define the integral of ϕ over $\mathrm{GL}_n(F)$ is defined by

$$\int_{GL_n(F)} \phi(\tau)d\tau = \int_{M_N(F)} \phi(x)|\det(x)|^{-n}dx.$$

For any $g \in \mathcal{L}(\mathrm{GL}_n(E))$, we can define $g^0 : \mathrm{GL}_n(F) \to \mathbb{C}((X))$ by

$$g^0(x) = \begin{cases} g(\overline{x}) & x \in \mathrm{GL}_n(\mathcal{O}_F). \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.14. *The integral defined above has the following properties:*

(1) *The integral is well defined, which means that if $f_1, f_2 \in \mathcal{L}(M_n(F))$ are equal when restricted to $\mathrm{GL}_n(F)$, then $f_1 = f_2$.*

(2) *Suppose that $\phi \in \mathcal{L}(\mathrm{GL}_n(F))$ and $\sigma \in GL_n(F)$. Then $\tau \mapsto \phi(\tau\sigma)$ and $\tau \mapsto \phi(\tau\sigma)$ also belongs to $\mathcal{L}(GL_n(F))$, with*

$$\int_{GL_n(F)} \phi(\sigma\tau)d\tau = \int_{GL_n(F)} f(\tau)d\tau = \int_{\mathrm{GL}_n(F)} \phi(\tau\sigma)d\tau.$$

(3) *Suppose that g is a complex valued Schwartz-Bruhat function on $\mathrm{GL}_n(E)$ such that*

$$f(x) = \begin{cases} g(x)|\det x|^{-n} & x \in \mathrm{GL}_n(E) \\ 0 & \det x = 0 \end{cases}$$

is GL-Fubini on $M_n(E)$. Then g^0 belongs to $\mathcal{L}(GL_n(F))$, and

$$\int_{\mathrm{GL}_n(F)} g^0(\tau)d\tau = \int_{\mathrm{GL}_n(E)} g(u)du.$$

Proof. These follows from[Mor08, Remark 4.3, Proposition 4.4, Proposition 4.8]. □

Remark 2.15. Using these definitions of [Mor08], we find that

$$\sum c_k \prod_{i=1}^n \mu(a_i^k + t_2^{\gamma_i^k} t_1^{e_i^k} O^n) = \sum c_k \prod_{i=1}^n t_1^{-e_i^k} X^{\gamma_i^k} = \sum c_k \left((\prod \mu_E(t_1^{e_i^k} O_E)) X^{\sum \gamma_i^k} \right).$$

We then have

$$\sum c_k \prod_{i=1}^n \mu(a_i^k + t_2^{\gamma_i^k} t_1^{e_i^k} O^n) = \sum c_k \left((\int_{E^n} f_k d\mu_{E^n}) X^{\sum_i \gamma_i^k} \right).$$

Like the one-dimensional case, a distinguished set of F^n is of the form

$$\prod_{i=1}^n a_i + t_2^{\gamma_i} t_1^{e_i} O, \quad a_i, b_j \in \mathbb{Z},$$

and its measure is $\mu(\prod_{i=1}^n a_i + t_2^{\gamma_i} t_1^{e_i} O) = \prod_{i=1}^n \mu(a_i + t_2^{\gamma_i} t_1^{e_i} O) = \prod_{i=1}^n q^{-e_i} X^{\gamma_i}$. Let $\mathcal{A}_{F^n}^{\text{dist}}$ be the collection of distinguished sets and let $\mathcal{A}_{F^n}$ be the minimal ring generated by the distinguished set. Then for finitely many disjoint sets $A_k \in \mathcal{A}^{F^n}$, we have $\sum_{i=1}^N c_k A_k = \sum c_k \mu(A_k)$.

Remark 2.16. Generally, it is not easy to define distinguished sets of GL_n, since usually the measure on $GL_n(F)$ should be $|\det x|^{-n} d_{M_n(F)} x$, but the determinant is not usually a constant.

2.5. **Convolution product of integrable functions over** $\mathrm{GL}_n(F)$**.** For any $g : \mathrm{GL}_n(E) \to \mathbb{C}$, $A \in M_n(F), \Gamma = (\gamma_{i,j})_{1\leq i,j\leq n}$. Let $A + t_2^{\Gamma} M_n(\mathcal{O})$ denote set of matrices $(a_{i,j} + t_2^{\gamma_{i,j}}\mathcal{O})_{1\leq i,j\leq n}$,we can define $g^{A,\Gamma} : \mathrm{GL}_n(F) \to \mathbb{C}((X))$ by

$$g^{A,\Gamma}(x) = \begin{cases} g(\overline{(x-A)t_2^{-\Gamma}}) & x \in A + t_2^{\Gamma} M_n(\mathcal{O}) \text{ and } \det x \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

It is easy to find $g^0 = g^{0,0}$. Similarly for any $g : M_n(E) \to \mathbb{C}$, we can define $g^{A,\Gamma} : M_N(F) \to \mathbb{C}((X))$.

Definition 2.17. Let V be a subset of $\mathrm{GL}_n(F)$, we will say that V is a measurable subset if char_V belongs to $\mathcal{L}(\mathrm{GL}_n(F))$, i.e., $\mathrm{char}_V |\det x|^{-n}$ extends to a function in $\mathcal{L}(M_n(F))$.

Lemma 2.18. *For any two sets of the form $\tau(\prod_{1\leq i\leq n}(a_i + t_2^{\gamma_i}\mathcal{O}))$, $\tau \in \mathrm{GL}_n(F)$,their intersection is either empty or also has this form.*

Proof. Since in a two-dimensional local field F, if we have two sets $a + t_2^{\gamma_a}\mathcal{O}$ and $b + t_2^{\gamma_b}$, then their intersection is either empty or one contains the other. Thus for two sets of the form $\prod(a_i, +t_2^{\gamma_i}\mathcal{O})$ and $\prod(b_i + t_2^{\delta_i}\mathcal{O})$, their intersection is

(1) empty, if there is a i such that $a_i + t_2^{\gamma_i} \cap b_i + t_2^{\delta_i} = \emptyset$;
(2) $\prod c_i + t_2^{\xi_i}\mathcal{O}$, where $(c_i + t_2^{\xi_i}\mathcal{O})$ equals $a_i + t_2^{\gamma_i}\mathcal{O}$ or $b_i + t_2^{\delta_i}\mathcal{O}$.

Since for $\tau \in GL_n(F)$, we have $\tau(\prod_{1\leq i\leq n}(a_i + t_2^{\gamma_i}\mathcal{O}))$ also has the form $\prod_{1\leq i\leq n}(d_i + t_2^{\zeta_i}\mathcal{O})$, we get the desired conclusion. □

Lemma 2.19. *For $f_1, f_2 \in \mathcal{L}(F^n, GL_n(F))$, we have $f_1 \cdot f_2 \in \mathcal{L}(F^n, \mathrm{GL}_n(F))$.*

Proof. It is sufficient to prove for $f_1 = g_1^{A_1,\Gamma_1} \circ \tau_1, f_2 = g_2^{A_2,\Gamma_2} \circ \tau_2$. By the Lemma 2.18, we have either $\mathrm{Supp} f_1 \cap \mathrm{Supp} f_2 = \emptyset$ or $\mathrm{Supp} f_1 \cap \mathrm{Supp} f_2 = A + t_2^{\Gamma}\mathcal{O}^n = \prod_{1\leq i\leq n} a_i + t_2^{\gamma_i}\mathcal{O}$. If $\mathrm{Supp} f_1 \cap \mathrm{Supp} f_2 = \emptyset$, then $f_1.f_2 = 0$. If $\mathrm{Supp} f_1 \cap \mathrm{Supp} f_2 = \prod_{1\leq i\leq n} a_i + t_2^{\gamma_i}\mathcal{O}$, we have $g_1^{A_1,\Gamma_1} \circ \tau_1|_{\mathrm{Supp} f_1 \cap \mathrm{Supp} f_2} = g_1^{A,\Gamma}$ and $g_2^{a_2,\Gamma_2} \circ \tau_2|_{\mathrm{Supp} f_1 \cap \mathrm{Supp} f_2} = g_2^{A,\Gamma}$, thus $f_1 \cdot f_2 = (g_1 \cdot g_2)^{A,\Gamma}$. □

Corollary 2.20. *For $f_1, f_2 \in \mathcal{L}(M_n(F))$, we have $f_1 \cdot f_2 \in \mathcal{L}(M_n(F))$.*

Lemma 2.21. *Let g be a function on $\mathrm{GL}_n(E)$ such that*

$$f(x) = \begin{cases} g(x)|\det x|^{-n} & x \in \mathrm{GL}_n(E) \\ 0 & \det x = 0 \end{cases}$$

is GL-Fubini on $M_N(E)$. For any $A \in M_n(F), \Gamma \in M_n(\mathbb{Z}), \sigma \in \mathrm{GL}_n(F)$ such that $A + t_2^{\Gamma} M_n(\mathcal{O}) \cap \mathrm{GL}_n(F)$ be a measurable set, we have $g^{A,\Gamma} \circ r_\sigma \in \mathcal{L}(\mathrm{GL}_n(F))$.

Proof. Since $\int_{\mathrm{GL}_n(F)}$ is translation invariant, it is sufficient to prove the proposition for $g^{A,\Gamma}$. We have

$$g^{A,\Gamma}(x) = \begin{cases} g(\overline{(x-A)t_2^{-\Gamma}}) & x \in (a_{i,j} + t_2^{\gamma_{i,j}}\mathcal{O})_{1\leq i,j\leq n} \text{ and } \det x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

To prove that it belongs to $\mathcal{L}(\mathrm{GL}_n(F))$, we must have $g^{A,\Gamma}(x)|\det x|^{-n}$ extends to a function in $\mathcal{L}(M_n(F))$. We already have $g^{A,\Gamma}$ belongs to $\mathcal{L}(M_n(F))$, and $\mathrm{char}_{\mathrm{Supp}_{g^{A,\Gamma}}} |\det x|^{-n}$ also belongs to $\mathcal{L}(M_n(F))$. The results follows from Corollary 2.20.

□

We know that Bruaht-Schwartz functions are GL-Fubini functions. We now give an intermediate space of our study.

Definition 2.22. We let $\mathcal{L}^S(\mathrm{GL}_n F)$ denote the the $\mathbb{C}((X))$- subspace of $\mathcal{L}(\mathrm{GL}_n(F))$ consists of functions $f : \mathrm{GL}_n(F) \to \mathbb{C}((X))$ generated by

$$g^{A,\Gamma} \circ r_\sigma, \quad A \in M_n(F), \Gamma = (\gamma_{i,j})_{1\le i,j\le n}$$

which satisfies

(1) g is a complex valued Bruhat-Schwartz function on $\mathrm{GL}_n(E)$,
(2) $f(x) = \begin{cases} g(x)|\det x|^{-n} & x \in \mathrm{GL}_n(E) \\ 0 & \det x = 0 \end{cases}$ is GL-Fubini on $M_n(E)$, and
(3) the support of $g^{A,\Gamma} \circ r_\sigma$ is a measurable subset.

Lemma 2.23. *For any $f_1, f_2 \in \mathcal{L}^S(\mathrm{GL}_n(F)$, $f_1 \cdot f_2 \in \mathcal{L}^S(\mathrm{GL}_n(F))$.*

Proof. Without loss of generality, we may suppose that $f_1 = g_1^{A_1,\Gamma_1} \circ r_{\sigma_1}, f_2 = g_2^{A_2,\Gamma_2} \circ r_{\sigma_2}$. By the proof of Lemma 2.19, we see that $f_1 \cdot f_2 = g'^{A',\Gamma'}$, since we have Bruhat-Schwartz are compactly generated functions and so their product, we get $f_1 \cdot f_2 \in \mathcal{L}^s(\mathrm{GL}_n(F))$. □

Definition 2.24. We let $\mathcal{L}^H(\mathrm{GL}_n(F))$ denote the spaces of functions $f : \mathrm{GL}_n(F) \to \mathbb{C}((X))$, such that $f(x), f(\frac{1}{x}), \mathrm{char}_{\mathrm{Supp} f} \in \mathcal{L}^S(\mathrm{GL}_n(F))$.

Definition 2.25. For every $f_1, f_2 \in \mathcal{L}^H(\mathrm{GL}_n(F))$, we define the convolution product of f_1 and f_2 to be

$$f_1 * f_2(y) := \int_{\mathrm{GL}_n(F)} f_1(yg^{-1}) f_2(g) dg.$$

Lemma 2.26. *For any $f_1, f_2 \in \mathcal{L}^H(\mathrm{GL}_n(F)$, $f_1 * f_2 \in \mathcal{L}^H(\mathrm{GL}_n(F))$.*

Proof. We may assume $f_1 = g_1^{A_1,\Gamma_1} \circ r_{\sigma_1}, f_2 = g_2^{A_2,\Gamma_2} \circ r_{\sigma_2}$. Then we have $g_1^{A_1,\Gamma_1} \circ r_{\sigma_1}(yx^{-1}) g_2^{A_2,\Gamma_2} \circ r_{\sigma_2}(x)$ belongs to $\mathcal{L}^s(\mathrm{GL}_n(F))$, and $|\det x|^{-n}$ also belongs to $\mathcal{L}^s(\mathrm{GL}_n(F))$. Then we have

$$\begin{aligned} f_1 * f_2(y) &:= \int_{\mathrm{GL}_n(F)} f_1(yg^{-1}) f_2(g) dg = \int_{M_n(F)} f_1(yx^{-1}) f_2(x) |\det x|^{-n} dx \\ &= \int_{M_n(F)} (g_1^{A_1,\Gamma_k} \circ r_{\sigma_1}(yx^{-1}))(g_2^{A_2,\Gamma_2} \circ r_{\sigma_2}(x))(\sum a_w g_w^{A_w,\Gamma_w} \circ r_{\sigma_w}(x)) dx \end{aligned}$$

We notice that $f_1(yx^{-1}) = f_1((y^{-1}x)^{-1})$ also belongs to $\mathcal{L}^S(\mathrm{GL}_n(F))$, and by the intersection properties of those $A + t_2^\Gamma M_N(O)$. The above formula equals

$$\begin{aligned} &\int_{F^{n^2}} \Big((\sum a_k g_k^{A_k,\Gamma_k} \circ r_{\sigma_k}) g_2^{A_2,\Gamma_2} \circ r_{\sigma_2} (\sum a_w g_w^{A_w,\Gamma_w} \circ r_{\sigma_w}) \Big) \circ T(x) dx \\ &= \sum_{w,k} a_{w,k} \int_{F^{n^2}} \Big(g_k^{A_{w,k},\Gamma_{w,k}} \circ r_{\sigma_{w,k}}(y^{-1}x) \cdot g_2^{A_{w,k},\Gamma_{w,k}} \circ r_{\sigma_{w,k}}(x) \cdot g_w^{A_w,\Gamma_w} \circ r_{\sigma_{w,k}}(x) \Big) \circ T(x) dx \\ &= \sum_{w,k} a_{w,k} |\det r_{\sigma,w}|^{-1} \left(\int_{E^{n^2}} g_k g_2 g_w \bar{T}(x) dx \right) X^{\sum_{i,j} (\Gamma_{w,k})_{i,j}} \end{aligned}$$

We have $\int_{F^{n^2}} \Big(g_k^{A_{w,k},\Gamma_{w,k}} \circ r_{\sigma_{w,k}}(y^{-1}x) \cdot g_2^{A_{w,k},\Gamma_{w,k}} \circ r_{\sigma_{w,k}}(x) \cdot g_w^{A_w,\Gamma_w} \circ r_{\sigma_{w,k}}(x) \Big) \circ T(x) dx$ is a Bruhat-Schwartz function with variable y by a following lemma, thus

$f_1 * f_2(y)$ belongs to $\mathcal{L}^S(\mathrm{GL}_n(F))$. Using the same way, we can prove that $f_1 * f_2(\frac{1}{y})$ and $\mathrm{char}_{\mathrm{Supp} f_1 * f_2}$ also belongs to $\mathcal{L}^S(\mathrm{GL}_n(F))$. $\square$

Lemma 2.27. *The function* $\int_{F^{n^2}} g_1^{A,\Gamma} \cdot r_{\sigma_1}(y^{-1}x) g_1^{A,\Gamma} \cdot r_{\sigma_1}(x) \circ T dx$ *with variable* y *can be written as* $g^{A,\Gamma}(y)$ *for some Bruhat-Schwartz function* g*, and* $A \in M_n(F)$ *and* $\Gamma \in M_n(\mathbb{Z})$*.*

Proof. We may assume $g = \mathrm{char}_K$ for some compact open $K \subset E$. Let $V = r_{\sigma_1}(A + t_2^{\Gamma} p^{-1} K)$

$$\int_{F^{n^2}} g_1^{A,\Gamma} \cdot r_{\sigma_1}(y^{-1}x) g_1^{A,\Gamma} \cdot r_{\sigma_1}(x) \circ T dx = \int_{F^{n^2}} \mathrm{char}_V(y^{-1}x)\mathrm{char}_V(x) \circ T dx$$

To make $\mathrm{char}_V(y^{-1}x)\mathrm{char}_V(x)$ nonzero, we must have $A \in t_2^{\Gamma} M_n(\mathcal{O})$. Moreover the integration is equal to $\mathrm{char}_{p^- K} = (\mathrm{char}_K^E)^{0.0}$. $\square$

Theorem 2.28. *There is an* $\mathbb{C}((X))$*-associative algebra structure on the space* $\mathcal{L}^H(\mathrm{GL}_n(F))$*, we denote the corresponding algebra by* $\mathcal{H}$*, and call it the Hecke algebra of* $\mathrm{GL}_n(F)$*.*

Proof. We have proved that the convolution product is well-defined. The addition is $\mathbb{C}((X))$-linear by definition. We need to check that the convolution product is associative. For $f_1, f_2, f_3 \in \mathcal{C}_m(G(F))$. We have $f_1 * (f_2 * f_3)(y) =$

$$\begin{aligned}
&= \int_{G(F)} f_1(yg^{-1}) \left(\int_{G(F)} f_2(gh^{-1}) f_3(h) dh \right) dg \\
&= \int_{F^{n^2}} f_1 \circ T_a(yg^{-1}) \left(\int_{F^{n^2}} f_2 \circ T_b(gh^{-1}) f_3 \circ T_b(h) |\det T_b h|^{-n} dh \right) |\det T_a g|^{-n} dg \\
&= \int_{F^{2n^2}} f_1 \circ T_a(yg^{-1}) f_2 \circ T_b(gh^{-1}) f_3 \circ T_b(h) |\det T_b h|^{-n} |\det T_a g|^{-n} dh dg
\end{aligned}$$

By the Fubini properties of our integral, we can reverse the order of integration.

$$\begin{aligned}
&= \int_{F^{n^2}} \left(\int_{F^{n^2}} f_1 \circ T_a(yg^{-1}) f_2 \circ T_a(gh^{-1}) |\det T_a g|^{-n} dg \right) f_3 \circ T_b(h) |\det T_b h|^{-n} dh \\
&= \int_{F^{n^2}} \left(\int_{F^{n^2}} f_1 \circ T(yg^{-1}) f_2 \circ T(gh^{-1}) |\det g|^{-n} dg \right) f_3 \circ T(h) |\det h|^{-n} dh \\
&= \int_{F^{n^2}} \left(\int_{F^{n^2}} f_1 \circ T(yh^{-1}(gh^{-1})^{-1}) f_2 \circ T(gh^{-1}) |\det Tg|^{-n} d(gh^{-1}) \right) f_3 \circ T(h) |\det h|^{-n} dh \\
&= \int_{F^{n^2}} \left((f_1 * f_2) \circ T(yh^{-1}) \right) f_3 \circ T(h) |\det h|^{-n} dh \\
&= (f_1 * f_2) * f_3(y).
\end{aligned}$$

$\square$

3. $\mathbb{C}((X))$-representations of $\mathrm{GL}_n(F)$

3.1. Measurable representations of $\mathrm{GL}_n(F)$.

Definition 3.1. A measurable $\mathbb{C}((X))$-representation of $G(F)$ is a pair (V, π), where V is a $\mathbb{C}((x))$-vector space, and

$$\pi : \mathrm{GL}_n(F) \to \mathrm{Aut}_{\mathrm{Vect}_{\mathbb{C}((X))}}(V).$$

such that Stab_v is a measurable subgroup of G for every $v \in V$.

Given a representation $\pi : G \to \mathrm{GL}(V)$, for $f \in \mathcal{H}$, one obtains a linear map

$$\begin{aligned} \pi(f) : \qquad & V \to V \\ & v \mapsto \int_{\mathrm{GL}_n(F)} f(g)\pi(g) \cdot v dg. \end{aligned}$$

We obtain an action

$$\mathcal{H} \times V \to V.$$

Lemma 3.2. *The action $\mathcal{H} \times V \to V$ is an algebra action.*

Proof. we have

$$\begin{aligned} \pi(f_1 * f_2) \cdot v &= \int_{GL_n(F)} \left(\int_{GL_n(F)} f_1(gh^{-1}) f_2(h) dh \right) \pi(g) \cdot v dg \\ &= \int_{GL_n(F)} \left(\int_{GL_n(F)} f_1(g') f_2(h) dh \right) \pi(g'h) \cdot v d(g') \\ &= \int_{\mathrm{GL}_n(F)} \left(\int_{\mathrm{GL}_n(F)} f_1(g')\pi(g') f_1(h)\pi(h) \cdot v dh \right) dg'h \\ &= \int_{\mathrm{GL}_n(F)} f_1(g')\pi(g') \left(\int_{\mathrm{GL}_n(F)} f_2(h)\pi(h) \cdot v dh \right) dg'h \\ &= \int_{\mathrm{GL}_n(F)} f_1(gh^{-1})\pi(gh^{-1}) \left(\int_{\mathrm{GL}_n(F)} f_2(h)\pi(h) \cdot v dh \right) dg \\ &= \pi(f_1)(\pi(f_2) \cdot v), \end{aligned}$$

where the last equality comes from the translation property of the integral. □

3.2. **Representation on function spaces.** Suppose that we have a $\mathcal{L}^H(\mathrm{GL}_n(F))$-module V, how do we get a measurable representation? We first consider the action of G on $\mathcal{L}^H(\mathrm{GL}_n(F))$ by

$$(h \cdot f)(x) = f(h^{-1}x).$$

Proposition 3.3. *$\mathcal{L}^H(\mathrm{GL}_n(F))$ is a measurable representation of $\mathrm{GL}_n(F)$.*

Proof. By the translation invariant properties of the two-dimensional integral, we have $f(h^{-1}x)$ belongs to $\mathcal{L}^H(\mathrm{GL}_n(F))$. To prove it is a measurable representation, we need to prove that Stab_f is a measurable subgroup of $\mathrm{GL}_n(F)$. Since $\mathcal{L}^H(\mathrm{GL}_n(F))$ is generated by $g^{A,\Gamma} \circ r_\sigma$, where g is a Bruhat-Schwartz function on E. We may suppose that $g = \mathrm{char}_K^E$ for a compact subset of E, and $f = g^{A,\Gamma} \circ r_\sigma$. We then have

$$g^{A,\Gamma} \circ r_\sigma(x) = \mathrm{char}_V, \quad V = \left\{ x \in \mathrm{GL}_n(F) | r_\sigma x \in (A + t_2^{\gamma_{i,j}} p^{-1} K)_{1 \le i,j \le n} \right\},$$

$$g^{A,\Gamma} \circ r_\sigma(h^{-1}x) = \mathrm{char}_U, \quad U = \left\{ x \in \mathrm{GL}_n(F) | r_\sigma h^{-1} x \in (A + t_2^{\gamma_{i,j}} p^{-1} K)_{1 \le i,j \le n} \right\}.$$

By the assumption of f,

$$\mathrm{Stab}_f = \left\{ h \in \mathrm{GL}_n(f) | V = U \right\}.$$

That is
$$\mathrm{Stab}_f = \mathrm{Stab}_{r_\sigma^{-1}A} \cap \mathrm{Stab}_{p^{-1}K}.$$
Since K is a compact subset of E, for every $y \in K$, we have a compact open neighbourhood $K_y y \subset K$. Then
$$\{K_y y | y \in K\}$$
is an open cover of K, it has finite sub-cover $\{K_1 y_1, \cdots, K_s y_s\}$. Let
$$H = \cap_{1 \le i \le s} K_i.$$
Then K is left H invariant. We claim that $p^{-1}K$ is left $p^{-1}H$ invariant.

For $w \in p^{-1}H$, and $z \in p^{-1}K$, we have
$$p(wz) = p(w)p(z) \subset K,$$
so $wz \subset p^{-1}K$. On the other hand, we have $\mathrm{Stab}_{r\sigma^{-1}A} = e$. Thus $\mathrm{Stab}_f = \{e\}$ if $A \notin (t_2^{\gamma_{i,j}} p^{-1}K)_{1 \le i,j \le n}$, and $\mathrm{Stab}_f = p^{-1}H$ if $A \in (t_2^{\gamma_{i,j}} p^{-1}K)_{1 \le i,j \le n}$. We have e is a measure 0 subgroup, and
$$\int_{M_N(F)} \mathrm{char}_{p^{-1}H} |\det x|^{-n} dx = \int_{M_n(F)} (\mathrm{char}_H)^{0,0}(x) |\det x|^{-n} dx.$$
By [Mor08, Poposition 4.7], $p^{-1}H$ is a measurable subgroup. □

Lemma 3.4. *Every element $f \in \mathcal{L}^H(\mathrm{GL}_n(F))$ is right $\mathcal{L}^H(\mathrm{GL}_n(F)//M)$ for some measurable subgroup M of $\mathrm{GL}_n(F)$.*

Proof. We may suppose that $f = g^{0,\Gamma} \circ r_\sigma$, otherwise we can take $M = I_n$. Since g is a Bruhat-Schwartz function on E, we may suppose that $g = \mathrm{char}_K^E \in C_C^\infty(\mathrm{GL}_n(E))$ for a compact open subset $K \subset E$. Like the previous lemma, for every $y \subset K$, we can choose a compact open subgroup $K_y \subset E$, such that $yK_y \subset K$. We have
$$\{yK_y | y \in K\}$$
is an open cover of K, it has finite sub-cover $\{y_1 K_1, \cdots, y_s K_s\}$. Let
$$H_r = \cap_{1 \le i \le s} K_i.$$
Then K is right H_r-invariant. Similarly, we can find H_l such that K is left H_l invariant. Let
$$H = H_r \cap H_l,$$
we see that g is bi-K-invariant. Let $M = p^{-1}(H)$, we claim that $f = g^{0,\Gamma} \circ r_\sigma$ is bi-M-invariant. Suppose that $w_1, w_2 \in p^{-1}(H), z \in p^{-1}(K)$, we then have
$$p(w_1 z w_2) = p(w_1)p(z)p(w_2) \subset K,$$
thus $w_1 z w_2 \subset p^{-1}K$. We have
$$g^{0,\Gamma} \circ r_\sigma(x) = \mathrm{char}_V, \quad V = \left\{x \in \mathrm{GL}_n(F) | r_\sigma x \in (t_2^{\gamma_{i,j}} p^{-1}K)_{1 \le i,j \le n}\right\},$$
thus V is bi-M-invariant. We get $f \in \mathcal{L}^H(\mathrm{GL}_n(F)//M)$. □

Remark 3.5. Since $\mathcal{L}^H(\mathrm{GL}_n(F))$ is generated by $g^{a,\gamma} \circ r_\sigma$ and all g are compactly generated, since are locally constant functions. We then get for $f \in \mathcal{L}^H(GL_n(F)//M)$,
$$f = \sum f(g_i) \mathrm{Id}_{M g_i M}.$$
We notice that usually this sum has infinite items.

4. Two-dimensional adelic automorphic forms

4.1. Two Dimensional Adeles. In this subsection, we give a very short introduction to two-dimensional adeles, more details can be found in [Fes03] and [Fes10]. Let $B = \mathrm{Spec}\mathcal{O}_K$ for a number field K and let $\phi : X \to B$ be a B-scheme satisfying the following conditions:

(1) X is integral, regular and dimension 2.
(2) ϕ is proper and flat.
(3) The generic fibre X_K is a geometrically integral, smooth, projective curve over K.

Fix a closed point $x \in X$, and a curve $y \subset X$, such that $x \in y \subset X$. Let $p_{y,x} = \ker(\mathcal{O}_{X,x} \to \mathcal{O}_{y,x})$ and $\phi : \mathrm{Spec}\widehat{\mathcal{O}_{X,x}} \to \mathrm{Spec}\mathcal{O}_{X,x}$. For any $q \in \mathrm{Spec}\widehat{\mathcal{O}_{X,x}}$, such that $q \cap \mathcal{O}_{X,x} = p_{y,x}$, we call this q a local branches of y at x, denote all this q as $y(x)$. It can be proved that $K_{x,q} := \mathrm{Frac}(\widehat{(\mathcal{O}_x)_q})$ is a two-dimensional local field.

We make the following notations:

$$K_{x,y} := \prod_{q \in y(x)} K_{x,q}, \quad \mathcal{O}_{x,y} := \prod_{q \in y(x)} \mathcal{O}_{x,q}$$

$$E_{x,y} := \prod_{q \in y(x)} E_{x,q}, \text{ where } E_{x,q} \text{ is the residue field of } K_{x,q}.$$

$$k_{x,y} := \prod_{q \in y(x)} k_{x,q}, \text{ where } k_{x,q} \text{ is the residue field of } E_{x,q}.$$

For a nonsingular curve y on S and integer r, we define an adelic space

$$\mathbb{A}_y^r = \left\{ \sum_{i \geq r} a_i t_y^i | \text{ where } a_i \text{ are lifts of } \overline{a_i} \in \mathbb{A}_{k(y)} \text{to } a_i \in \mathcal{O}_{\substack{x,y \\ x \in y}} \right\}.$$

Definition 4.1. For a curve $y \subset S$, we have the following definitions, see [Fes08] for details.

(1) $\mathbf{A}_y := \cup_{r \in \mathbb{Z}} \mathbb{A}_y^r$ and $\mathbf{OA}_y := \mathbb{A}_y^0$.
(2) $\mathbb{A}_y := \mathbb{A}_y^0$, and $O\mathbb{A}_y := \mathbb{A}_y \cap \prod O_{x,y}$.
(3) $\mathbf{A} := \prod'_{y \subset S} \mathbf{A}_y$, the restrict product of $\mathbf{A}_y$ with respect to $\mathbf{OA}_y$.
(4) $\mathbb{A} := \prod'_{y \subset S} \mathbb{A}_y$, the restrict product of $\mathbb{A}_y$ with respect to $O\mathbb{A}_y$.
(5) $\mathbf{B} := \mathbf{A}_S \cap \prod K_y$, $\mathbf{C} := \mathbf{A} \cap \prod K_x$.
(6) $\mathbb{B} := \mathbb{A}_S \cap \prod \mathcal{O}_y$.

There is a commutative diagram

$$\begin{tikzcd}
& \mathbb{A}^\times \otimes \mathbf{A}^\times/\mathbf{VA}^\times \arrow[d] \arrow[dr] & \\
T \arrow[r] & \mathbb{A}^\times \times \mathbb{A}^\times/\mathbb{VA}^\times \arrow[r] & J/VJ
\end{tikzcd}$$

4.2. Automorphic forms on two-dimensional adeles. Let $G = \mathrm{GL}_n$ and let $\mathbb{T}_G = G(\mathbb{A}) \times G(\mathbb{A})/V(G(\mathbb{A}) \times G(\mathbb{A}))$. Let $\mathbb{K}_G$ be the image of the map

$$G(\mathbb{B}) \times G(K) \to G(\mathbb{A}) \times G(\mathbb{A})/G(\mathbb{A}) \to \mathbb{T}_G.$$

It was suggested by Fesenko that the two-dimensional analogue of $G(O\mathbb{A}_k)\backslash G(\mathbb{A}_k)/G(k)$ is

$$\mathbb{T}_G/\mathbb{K}_G.$$

In following content, we assume K is positive characteristic, then $V(\mathbb{A}) = O\mathbb{A}$. The two-dimensional adelic space can be viewed as

$$G(\mathbb{B}) \times G(K) \setminus G(\mathbb{A}) \times G(\mathbb{A})/G(O\mathbb{A}) \times G(O\mathbb{A}).$$

We let $\mathcal{L}^H(\mathrm{GL}_n(\mathbb{A}) \times \mathrm{GL}_n(\mathbb{A}))$ denote the functions on $\mathrm{GL}_n(\mathbb{A}) \times \mathrm{GL}_n(\mathbb{A})$ generated by $\otimes_{x\in y\subset S}(f^1_{x,y}, f^2_{x,y})$ where $f^i_{x,y} = \otimes_{z\in y(x)} f^i x, z \in \prod_{z\in y(x)} \mathcal{L}^H(\mathrm{GL}_n(K_{x,z}))$. Then for any $f \in \mathcal{L}^H(\mathrm{GL}_n(\mathbb{A}) \times \mathrm{GL}_n(\mathbb{A}))$, the integral of f on $\mathrm{GL}_n(\mathbb{A}) \times \mathrm{GL}_n(\mathbb{A})$ is defined to be

$$\int_{\mathrm{GL}_n(\mathbb{A})\times\mathrm{GL}_n(\mathbb{A})} f d\mu = \prod_{z\in y(x), x\in y\subset S} \int_{GL_n(K_{x,z})\times \mathrm{GL}_n(K_{x,z})} f^1_{x,z} \otimes f^2_{x,z}.$$

Having the integration on two-dimensional adelic space, we can have the concept of measurable subsets, measurable representations and so on. We have the following definitions.

Definition 4.2. A function $f : \mathbf{G}(\mathbb{A}) \times \mathbf{G}(\mathbb{A}) \to \mathbb{C}((X))$ is called an automorphic function if it satisfies the following conditions:

(1) $f \in \mathcal{L}^H(G(\mathbb{A}) \times G(\mathbb{A}))$;
(2) $f(\gamma g) = f(g)$, $\forall \gamma \in G(\mathbb{B}) \times G(K)$ and $\forall g \in G(\mathbb{A}) \times G(\mathbb{A})$.
(3) For any $v \in G(\mathbb{A}) \times G(\mathbb{A})$, $\{f(gv)|g \in G(\mathbb{A}) \times G(\mathbb{A})\}$ is a measurable representation of $G(\mathbb{A}) \times G(\mathbb{A})$.

Definition 4.3. Let $\mathcal{H}$ be the complex vector space of all smooth functions $\phi : G(\mathbb{A}) \times G(\mathbb{A}) \to \mathbb{C}((X))$. We define the convolution product on $\mathcal{H}$ by

$$\Phi_1 * \Phi_2 : g \mapsto \int_{G(\mathbb{A})\times G(\mathbb{A})} \Phi_1(gh^{-1})\Phi_2(h)dh$$

for $\Phi_1, \Phi_2 \in \mathcal{H}$. The complex vector space $\mathcal{H}$ together with the convolution product is called the Hecke algebra for $G(\mathbb{A} \times G(\mathbb{A}))$.

The Hecke algebra $\mathcal{H}$ acts on a continuous function $f : G(\mathbb{A}) \times G(\mathbb{A}) \to \mathbb{C}((X))$ by

$$\Phi(f) : g \mapsto \int_{G(\mathbb{A})\times G(\mathbb{A})} \phi(h)f(gh)dh.$$

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